PLANAR AND HAMILTONIAN COVER GRAPHS

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Noah Sametz Streib

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PLANAR AND HAMILTONIAN COVER GRAPHS

Approved by:

Dr. William T. Trotter, Advisor
School of Mathematics
Georgia Institute of Technology

Dr. Dwight Duffus
Mathematics & Computer Science
Department
Emory University

Dr. Prasad Tetali
School of Mathematics and
School of Computer Science
Georgia Institute of Technology

Dr. Robin Thomas
School of Mathematics
Georgia Institute of Technology

Dr. Joel Sokol
H. Milton Stewart School of Industrial
and Systems Engineering
Georgia Institute of Technology

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To my parents, Lynn and Victor,

for all of their encouragement and support,

and to my wife Amanda,

without whom none of this would have been possible.

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SUMMARY

This dissertation has two principal components: the dimension of posets with planar cover graphs, and the cartesian product of posets whose cover graphs have hamiltonian cycles that parse into symmetric chains. Posets of height two can have arbitrarily large dimension. In 1981, Kelly provided an infinite sequence of planar posets that shows that the dimension of planar posets can also be arbitrarily large. However, the height of the posets in this sequence increases with the dimension. In 2009, Felsner, Li, and Trotter conjectured that for each integer $h \geq 2$, there exists a least positive integer c_h so that if P is a poset with a planar cover graph (the class of posets with planar cover graphs includes the class of planar posets) and the height of P is h, then the dimension of P is at most c_h. In the first principal component of this dissertation we prove this conjecture. We also give the best known lower bound for c_h, noting that this lower bound is far from the upper bound. In the second principal component, we consider posets with the Hamiltonian Cycle–Symmetric Chain Partition (HC-SCP) property. A poset of width w has this property if its cover graph has a hamiltonian cycle which parses into w symmetric chains. This definition is motivated by a proof of Sperner's theorem that uses symmetric chains, and was intended as a possible method of attack on the Middle Two Levels Conjecture. We show that the subset lattices have the HC-SCP property by showing that the class of posets with the strong HC-SCP property, a slight strengthening of the HC-SCP property, is closed under cartesian product with a two-element chain. Furthermore, we show that the cartesian product of any two posets from this class has the HC-SCP property.

CHAPTER I

INTRODUCTION

The two principal components of this dissertation, Chapters 2 and 3, study fundamental combinatorial properties of a poset's cover graph; the first component studies posets with planar cover graphs, and the second studies posets whose cover graphs are hamiltonian. However, the majority of the definitions and terminology used within each component is disjoint. Accordingly, we have chosen to provide the definitions, notation, and background specific to the first component within the first components' introduction, and similarly for the second component. The purpose of this chapter is to ensure that the reader is familiar with the definitions common to both components and the notation that will be used throughout the dissertation. We further provide a brief introduction to the central parameters of this dissertation.

1.1 Basic definitions and notation

A *partially ordered set* or *poset* $\mathbf{P}$ is a pair (X, P) where X is a set and P is a reflexive, antisymmetric, and transitive binary relation on X. We call X the *ground set* while P is a *partial order* on X. Elements of the ground set X are also called *points*, and the poset $\mathbf{P}$ is *finite* if its ground set X is a finite set. In this dissertation, we will always assume that the ground set of a poset is finite.

When $\mathbf{P} = (X, P)$ is a poset, it is common to write $x \leq y$ in P and $y \geq x$ in P when $(x, y) \in P$. Of course, the notations $x < y$ in P and $y > x$ in P mean $x \leq y$ in P and $x \neq y$. When the poset $\mathbf{P}$ remains fixed throughout a discussion, we will sometimes abbreviate $x \leq y$ in P by just writing $x \leq y$, etc. When x and y are distinct points from X, we say x is *covered* by y in P (or y covers x in P) when $x < y$ in P, and there is no point $z \in X$ for which $x < z$ an $z < y$ in P. We can then

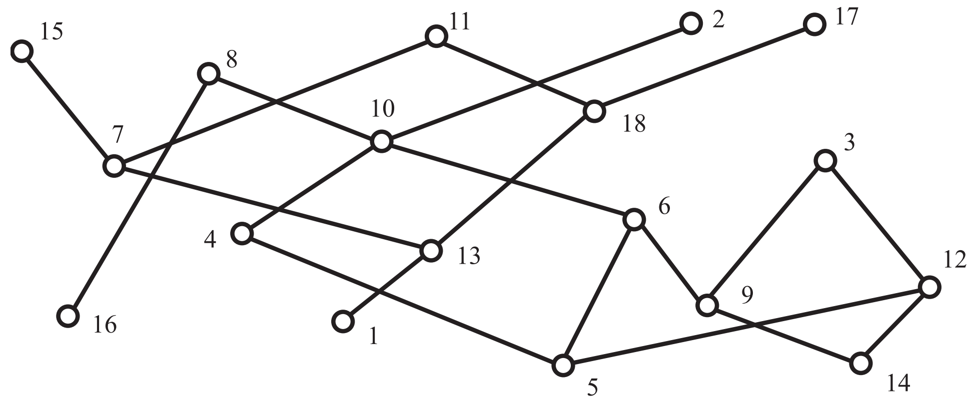


Figure 1.1: An order diagram

associate with the poset $\mathbf{P}$ a *cover graph* $\mathbf{G}$ whose vertex set is the ground set X of $\mathbf{P}$ with xy an edge in $\mathbf{G}$ if and only if one of x and y covers the other in $\mathbf{P}$. Cover graphs will be of particular interest to us in this dissertation. In particular, we will be able to study posets by using graph theoretic tools to examine their cover graphs.

It is convenient to illustrate a poset with a suitably drawn diagram of the cover graph in the Euclidean plane. In particular, we choose to require that the vertical coordinate of the point corresponding to y be larger than the vertical coordinate of the point corresponding to x whenever y covers x in P. Such diagrams are called *Hasse diagrams*, *poset diagrams*, *order diagrams*, or just *diagrams*. Figure 1.1 exhibits an order diagram, which is of course also a drawing of the cover graph of the same poset.

We also assume basic familiarity with the following terms: comparable, incomparable, chain, antichain, height, width, maximum element, minimum element, upset, and downset. We illustrate these concepts with an example. Consider the poset $\mathbf{P} = (X, P)$ described by the order diagram in Figure 1.1. We see 4 is comparable to 2, with $4 <_P 2$. Moreover, 4 is not covered by 2 but instead covered by 10. The elements 4 and 6 are incomparable in P, written $4 \| 6$. The elements $\{14, 9, 6, 8\}$ are a chain, the elements $\{7, 10, 18, 3\}$ are an antichain, the height of $\mathbf{P}$ is five, and the width of $\mathbf{P}$ is seven. The elements 3 and 11 are maximal and the elements 16 and 5 are minimal. The upset of 4 is $U(4) = \{2, 8, 10\}$, while we define the closed upset to be $U[4] = \{2, 4, 8, 10\}$. Similarly, the downset of 4 and the closed downset of 4 are $D(4) = \{5\}$ and $D[4] = \{4, 5\}$, respectively. When we refer to the *length* of a path in a cover graph, we mean the number of vertices on that path (so, in this dissertation, the height of a chain is equal to its length).

One of the most classic theorems in the theory of partially ordered sets is the following theorem of Dilworth [14].

Theorem 1.1.1 (Dilworth)**.** *Let* $\mathbf{P} = (X, P)$ *be a poset with width* w*. Then there exist* w *disjoint chains* $C_1, C_2, \ldots, C_w$ *such that* $X = C_1 \cup C_2 \cup \cdots \cup C_w$.

Theorem 1.1.1 belongs to a broad class of combinatorial theorems for which an obvious necessary condition is in fact sufficient. Other examples include Hall's theorem, Tutte's 1-factor theorem and Menger's theorem.

Throughout this dissertation, for $n \geq 1$, the set $\{1, 2, \ldots, n\}$ will be denoted $[n]$.

1.2 Dimension and planarity

A central concept in the combinatorics of finite posets is a parameter called *dimension*. We will delve more fully into dimension in Chapter 2, so for now we will be content to simply develop intuition. When $\mathbf{P} = (X, P)$ is a poset, a *linear extension* L of P is a total order on X such that $x <_P y$ implies that $x <_L y$. In a seminal 1941 paper, Dushnik and Miller [15] showed that every partial order P is the intersection of a collection of linear extensions of P, and defined the *dimension* of $\mathbf{P}$, $\dim(\mathbf{P})$, to be the size of the smallest such collection. For example, the three linear extensions below of the poset in Figure 1.2 show that its dimension is at most three:

$$L_1: \ b < e < a < d < g < c < f$$

$$L_2: \ a < c < b < d < g < e < f$$

$$L_3: \ a < c < b < e < f < d < g$$

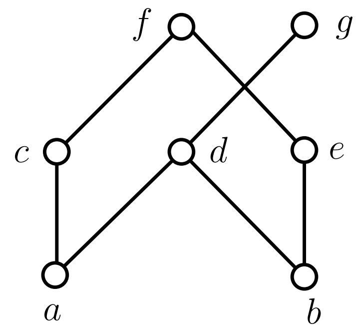


Figure 1.2: A 3-dimensional poset

It is a simple exercise to show that this poset has dimension equal to three. In fact, if we remove g from the poset we are left with a smaller 3-dimenional poset, but any further deletion will result in a 2-dimensional poset.

From a computational perspective, dimension is a complex parameter, since the decision problem: "Given a poset $\mathbf{P}$ and a natural number k, is $\dim \mathbf{P} \leq k$?" is NP-complete for $k \geq 3$ [56]. In fact, the same question is NP-complete for the class of height two posets when $k \geq 4$. Determining the computational complexity for height two posets when $k = 3$ remains an open problem. Approximation is also hard; approximating the dimension of an n-element poset in polynomial time within a factor of $\sqrt{n}$ would imply that NP = ZPP [25]. (The complexity class ZPP contains problems for which there is a probabilistic Turing machine that runs in polynomial time and either returns the correct answer or says "Do Not Know." It is known that P $\subseteq$ ZPP, and many computer scientists believe that P = ZPP.)

There are many analogies between the dimension of a poset and the chromatic number of a graph. In fact, the two main complexity results above were proven using reductions from graph colorability problems. However, with respect to planarity, some of these analogies break down. One reason for this break is that the class of planar graphs is closed under taking subgraphs, whereas the class of planar posets is not closed under taking subposets. This fact has the following consequences:

(1) There are linear-time algorithms for testing graph planarity [27], yet testing poset planarity is NP-complete [23].

(2) It is well-known from the Four Color Theorem [2, 3, 4, 42] that all planar graphs are 4-colorable, and for a while it was thought that all planar posets have dimension at most four, yet this is not true [31].

In Chapter 2 we turn our attention to the dimension of planar posets. While we cannot hope to bound their dimension in general, we show that the dimension can be bounded as a function of height.

For well over 30 years, the study of the combinatorics of posets has focused in large part on dimension theory. Readers seeking additional background material may find it helpful to consult Trotter's monograph [48] and survey article [49].

1.3 Hamiltonian cycles in cover graphs

Finding a cycle that uses all of the vertices in a graph is a classical problem in graph theory. In fact, both the undirected and directed hamiltonian cycle problems were in Karp's 21 NP-complete problems [29]. Shortly thereafter, Garey and Johnson showed that the both problems remain NP-complete even when the class of graphs in the input is greatly restricted; cubic graphs for the undirected problem and planar graphs for the directed problem [21, 22]. So, it is of interest to determine conditions that imply that a class of graphs is hamiltonian.

It is well-known that the cover graphs of subset lattices are hamiltonian. In Chapter 3 we strengthen this by combining it with Dilworth's theorem — we show that the cover graph of a subset lattice has a hamiltonian cycle that parses into w chains, where w is the width of the lattice. In fact, we obtain this result from a more general treatment involving the cartesian product of posets.

CHAPTER II

DIMENSION FOR POSETS WITH PLANAR COVER GRAPHS

2.1 Introduction

In this chapter, we focus on combinatorial problems associated with order diagrams and cover graphs. On the left side of Figure 2.1, we show the order diagram of a poset **P** on eight points. An order diagram is a drawing of the cover graph—but with restrictions on the locations of points. In the middle of this figure, we show another drawing of the cover graph of **P**, while on the right side of the figure, we show a drawing of the comparability graph of **P**.

In some sense, it is easy to characterize graphs that are cover graphs, as we have the following self-evident proposition.

Proposition 2.1.1. *A graph G is a cover graph if and only if the edges of G can be oriented so that there are no oriented paths $x_1, x_2, x_3, \ldots, x_n$ where $n \geq 3$ and x_1x_n is an edge in G.*

Nevertheless, it is quite difficult to devise an algorithm for implementing this test; in fact, Nešetřil and Rödl [38] and Brightwell [10] have shown that answering whether

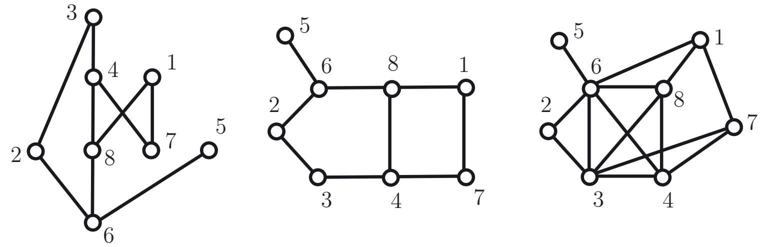


Figure 2.1: A poset with its cover and comparability graph

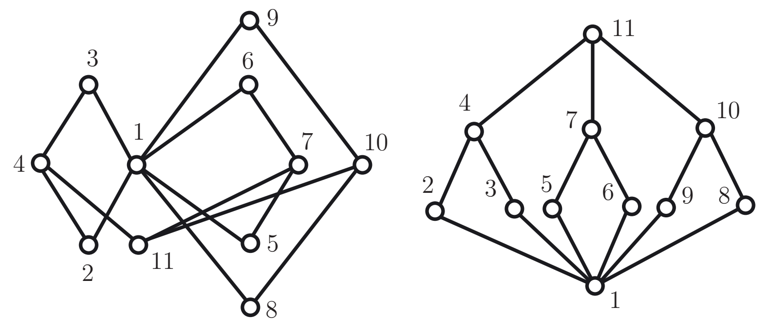


Figure 2.2: A non-planar poset having a planar cover graph

a graph is a cover graph is NP-complete.

A poset $\mathbf{P}$ is said to be *planar* if it has an order diagram without edge crossings. The poset shown in Figure 2.1 is planar even though the order diagram shown has edge crossings. Clearly, this diagram can be redrawn so that edges do not cross.

If a poset is planar, then its cover graph is planar, but the converse need not be true. On the left side of Figure 2.2, we show the diagram of a non-planar poset. On the other hand, as evidenced by the drawing on the right side of this figure, this poset does have a planar cover graph.

As is well known, there are very fast algorithms for testing graph planarity, in fact, with running time linear in the number of edges [27]. On the other hand, Garg and Tamassia [23] showed that it is NP-complete to answer whether a poset is planar.

The *dimension* of a partially ordered set $\mathbf{P}$, denoted $\dim(\mathbf{P})$, is the least positive integer t for which there are linear orders $L_1, L_2, \ldots, L_t$ on the ground set of $\mathbf{P}$ so that $P = L_1 \cap L_2 \cap \cdots \cap L_t$, where P is the partial order on the ground set. We assume basic familiarity with the notion of dimension, although we will provide in Section 2.4 a concise review of essential topics and techniques.

An element of a poset is called a *one* when it is the unique maximal element. Dually, a *zero* in a poset is the unique minimal element. The following result is due to C. R. Platt [39].

Theorem 2.1.2. *Let* $\mathbf{P}$ *be a finite lattice. Then* $\mathbf{P}$ *is planar if and only if the graph obtained from the cover graph of* $\mathbf{P}$ *by adding an edge between the zero and the one is a planar graph.*

In a similar direction is the following result that appears as an exercise in Birkhoff [7], where it is credited to Zilber.

Theorem 2.1.3. *Let* $\mathbf{P}$ *be a finite lattice. Then* $\mathbf{P}$ *is planar if and only if it has dimension at most* 2.

Even more is true, and while the following extension may be considered part of the folklore of the subject, certainly most of the credit should be given to Baker, Fishburn, and Roberts [5].

Theorem 2.1.4. *Let* $\mathbf{P}$ *be a finite poset with a one and a zero. Then* $\mathbf{P}$ *is planar if and only if* $\mathbf{P}$ *is a 2-dimensional lattice.*

Figure 2.3 shows a planar lattice with a zero and a one. The projections of the elements of the poset onto the horizontal and vertical axes give two linear extensions that realize the partial order.

If we relax the restriction on $\mathbf{P}$ having *both* a one and a zero we have the following theorem, due to Trotter and Moore [51].

Theorem 2.1.5. *Let* $\mathbf{P}$ *be a poset with a one (or a zero). If* $\mathbf{P}$ *is planar, then the dimension of* $\mathbf{P}$ *is at most* 3.

We show in Figure 2.4 three planar posets. Each has a one, and if the one is removed, the subposet remaining has dimension 3 and is irreducible, i.e., the removal of any point lowers the dimension to 2.

For $n \geq 2$, the *standard example* S_n is a height two poset with minimal elements $a_1, a_2, \ldots, a_n$, maximal elements $b_1, b_2, \ldots, b_n$, with $a_i < b_j$ in S_n if and only if $i \neq j$. To see that the dimension of S_n is at least n, notice that only one incomparable pair

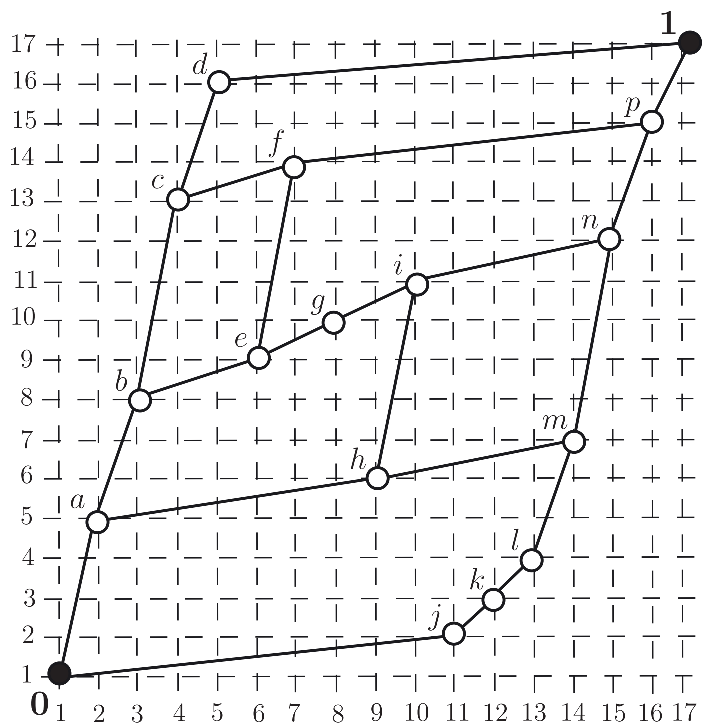


Figure 2.3: A planar poset with a zero and a one

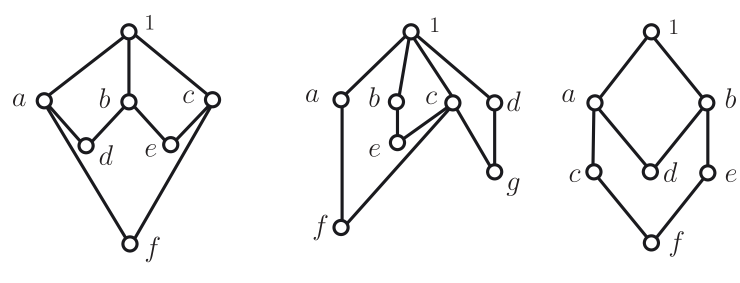


Figure 2.4: 3-dimensional planar posets with ones

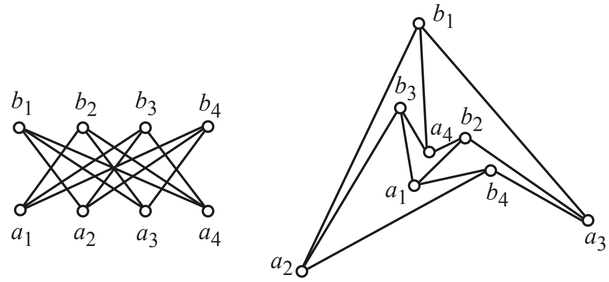


Figure 2.5: S_4 is planar

in the set $\{(a_i, b_i)\}_{i=1}^{n}$ can be reversed in any linear extension of S_n. For the upper bound, notice that any family of n linear extensions that reverses the incomparable pairs in the same set will in fact reverse all of the incomparable pairs in S_n. So the dimension of S_n equals n. Furthermore, S_n is irreducible if $n \geq 3$.

As evidenced in Figure 2.5, the standard example S_4 is planar, so there exist planar posets of dimension 4. For $n \geq 5$, the cover graph of the standard example S_n is non-planar. For a brief moment in time, it was believed that it might be true that $\dim(\mathbf{P}) \leq 4$ whenever $\mathbf{P}$ was planar, and that perhaps this inequality might even hold when the cover graph of $\mathbf{P}$ was planar. However, this appealing possibility unraveled. First, in [50], Trotter showed that there are posets of arbitrary dimension whose order diagram can be drawn without crossings on a sphere. Second, D. Kelly [31] showed that the standard example S_n is a subposet of a planar poset for all $n \geq 5$. We illustrate Kelly's construction in Figure 2.6, when $n = 5$, noting that the construction is easily generalized when $n \geq 6$.

2.2 Planar graphs and dimension

Kelly's construction temporarily put an end to explorations of links between planarity and dimension. But that situation changed dramatically with the breakthrough work of W. Schnyder [45], which provides a test for graph planarity in terms of the dimension of incidence posets. The *incidence poset* (also called the *vertex-edge* poset) of a

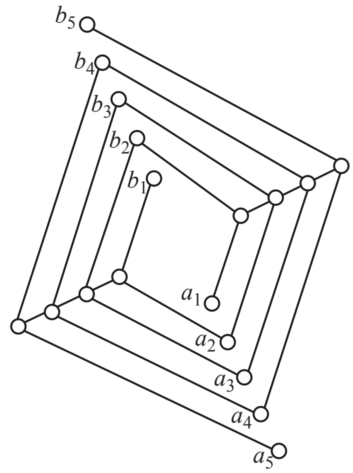


Figure 2.6: Kelly's construction

graph G is the height two poset P_G having the vertices of G as minimal elements, the edges of G as maximal elements, and $x < e$ in P_G if and only if x is an end of e in G.

Theorem 2.2.1. *Let G be a graph and let P_G be the incidence poset of G. Then G is planar if and only if* $\dim(P_G) \leq 3$.

The machinery developed by Schnyder in his proof of Theorem 2.2.1 has led to deep insights in other areas of mathematics, such as graph drawing (e.g. see [18]). Quite recently, Barrera-Cruz and Haxell [6] provided a shorter proof the same result that avoids the machinery in Schnyder's proof. Brightwell and Trotter [11] extended Schnyder's theorem with the following result.

Theorem 2.2.2. *Let* $\mathbf{P}$ *be the vertex-edge-face poset of a convex polytope in* $\mathbb{R}^3$. *Then* $\dim(\mathbf{P}) = 4$. *Furthermore, the subposet of* $\mathbf{P}$ *determined by the vertices and faces is* 4*-irreducible.*

In view of Steinitz's characterization of 3-connected planar graphs [47], the preceding theorem has the following alternative formulation.

Theorem 2.2.3. *Let* $\mathbf{P}$ *be the vertex-edge-face poset of a* 3*-connected planar graph drawn without edge crossings in the plane. Then* $\dim(\mathbf{P}) = 4$. *Furthermore, the* *subposet of* $\mathbf{P}$ *determined by the vertices and faces is* 4*-irreducible.*

S. Felsner [17] has provided an elegant and much shorter proof of Theorem 2.2.3. For general maps, with loops and multiple edges allowed, we have the following extension due to Brightwell and Trotter [12], with Theorem 2.2.3 serving as the base case in the inductive argument.

Theorem 2.2.4. *Let* $\mathbf{P}$ *be the vertex-edge-face poset of a planar multi-graph drawn without edge crossings in the plane. Then* $\dim(\mathbf{P}) \leq 4$.

In the general setting, we lose the tightness of the inequality as well as any notion of a specific irreducible subposet. Also, different drawings of the same multi-graph can produce vertex-edge-face posets having different values for dimension.

2.3 Posets having a planar cover graph

We show in Figure 2.7 a planar cover graph of a poset $\mathbf{P}$ that (1) has a one and (2) contains the standard example S_8. Again, this drawing is just one instance of an infinite family and shows that there is no analogue of Theorem 2.1.5 for cover graphs.

A poset of height 1 is an antichain, and non-trivial antichains have dimension 2. For height 2 posets, we have the following theorem proved by Felsner, Li and Trotter [19].

Theorem 2.3.1. *Let* $\mathbf{P}$ *be a poset of height* 2*. If the cover graph of* $\mathbf{P}$ *is planar, then* $\dim(\mathbf{P}) \leq 4$.

The standard example S_4 shows that the inequality in Theorem 2.3.1 is best possible. Also, we note that the proof of Theorem 2.3.1 proceeds by showing that $\mathbf{P}$ is isomorphic to the vertex-face poset of a planar map, so that the upper bound from Theorem 2.2.4 may be applied. Independent of this machinery, we know of no

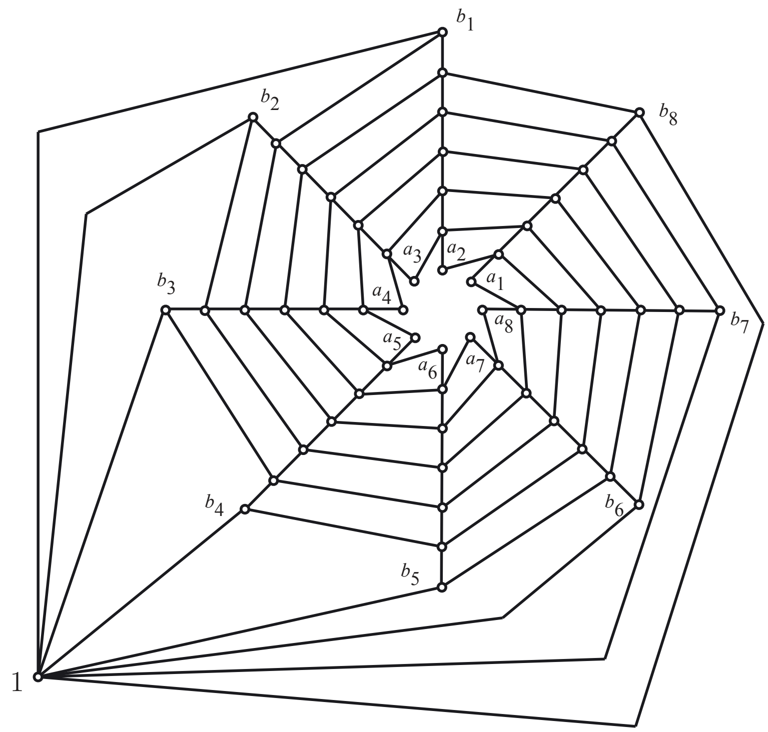


Figure 2.7: A poset with a one and a planar cover graph

entirely simple[1] argument to show that the dimension of a poset of height 2 having a planar cover graph is bounded—even by a very large constant. Furthermore, we do not see how the techniques developed in [19] can be extended to the case $h \geq 3$.

Before proceeding to the main theorem, we pause to provide a concise summary of the essential dimension theory background necessary for the arguments to follow. Readers who are familiar with the basic concepts and proof techniques of dimension theory can safely skip the material in the next section prior to Lemma 2.4.5.

[1]Felsner, Li and Trotter show that that the dimension of a poset of height two can be bounded as a function of the acyclic chromatic number of the cover graph. Since a planar graph has acyclic chromatic number at most six [1], this yields a bound on the dimension of the poset. However, this bound is 96, and while the argument can no doubt be tightened, it is unlikely to yield the correct answer which is four.

2.4 Dimension and alternating cycles

A family $\mathcal{R}$ of linear extensions of $\mathbf{P}$ is called a *realizer of* $\mathbf{P}$ if $\mathbf{P} = L_1 \cap L_2 \cap \cdots \cap L_t$, i.e., $x \leq y$ in $\mathbf{P}$ if and only if $x \leq y$ in L_i, for each $i = 1, 2, \ldots, t$. The *dimension* of $\mathbf{P}$, denoted $\dim(\mathbf{P})$, is then the least positive integer t for which $\mathbf{P}$ has a realizer of size t. Evidently, a poset has dimension 1 if and only if it is a chain, so in what follows, we will consider only posets that are not chains. For these posets, it useful to have a characterization of families of linear extensions that are realizers.

Let X denote the ground set of a poset $\mathbf{P}$ and let $\mathrm{Inc}(\mathbf{P}) = \{(x, y) \in X \times X : x \| y$ in $\mathbf{P}\}$, where $x \| y$ means that x and y are incomparable.

Proposition 2.4.1. *A family $\mathcal{R}$ of linear extensions of $\mathbf{P}$ is a realizer of $\mathbf{P}$ if and only if for every $(x, y) \in \mathrm{Inc}(\mathbf{P})$, there is some $L \in \mathcal{R}$ with $x > y$ in L.*

Let $(x, y) \in \mathrm{Inc}(\mathbf{P})$ and let L be a linear extension of $\mathbf{P}$. We say L *reverses* (x, y) if $y < x$ in L. A subset $S \subseteq \mathrm{Inc}(\mathbf{P})$ is *reversible* when there is a linear extension L reversing all pairs from S. Also, we say that a family $\mathcal{R}$ of linear extensions *reverses* S if for every $(x, y) \in S$, there is some $L \in \mathcal{R}$ with $y < x$ in L.

Proposition 2.4.2. *A family $\mathcal{R}$ of linear extensions of $\mathbf{P}$ is a realizer of $\mathbf{P}$ if and only if $\mathcal{R}$ reverses the set $\mathrm{Inc}(\mathbf{P})$ of incomparable pairs in $\mathbf{P}$.*

An incomparable pair $(x, y) \in \mathrm{Inc}(\mathbf{P})$ is called a *critical pair* of $\mathbf{P}$ when (1) $z < x$ in $\mathbf{P}$ implies $z < y$ in $\mathbf{P}$, for all $z \in X$, and (2) $w > y$ in $\mathbf{P}$ implies $w > x$ in $\mathbf{P}$, for all $w \in X$. We let $\mathrm{Crit}(\mathbf{P})$ denote the set of all critical pairs of $\mathbf{P}$. The following result [40] states that, in building a realizer, it suffices to reverse the critical pairs.

Proposition 2.4.3. *A family $\mathcal{R}$ of linear extensions of $\mathbf{P}$ is a realizer of $\mathbf{P}$ if and only if $\mathcal{R}$ reverses the set $\mathrm{Crit}(\mathbf{P})$ of critical pairs in $\mathbf{P}$.*

An *alternating cycle of length k* in $\mathbf{P}$ is a subset $S = \{(x_i, y_i) : 1 \leq i \leq k\} \subseteq \mathrm{Inc}(\mathbf{P})$ with $x_i \leq y_{i+1}$ in $\mathbf{P}$ for each $i = 1, 2, \ldots, k$ (here the subscripts are interpreted cyclically, i.e., $y_{k+1} = y_1$). An alternating cycle is *strict* when $x_i \leq y_j$ if and only if $j = i + 1$ (cyclically), for all $i, j = 1, 2, \ldots, k$.

The following elementary lemma from [51] is key to the results in this chapter.

Lemma 2.4.4. *Let $\mathbf{P}$ be a poset and let $S \subseteq \mathrm{Inc}(\mathbf{P})$.*

(1) *If S contains an alternating cycle, then it also contains a strict alternating cycle.*

(2) *S is reversible if and only if it does not contain an alternating cycle.*

(3) *S is reversible if and only if it does not contain a strict alternating cycle.*

Let $\min(\mathbf{P})$ and $\max(\mathbf{P})$ denote respectively the set of minimal elements and the set of maximal elements of a poset $\mathbf{P}$. For a poset $\mathbf{P}$, let $\mathrm{Crit}^*(\mathbf{P})$ denote the set $\{(x, y) \in \mathrm{Crit}(\mathbf{P}) \mid x \in \min(\mathbf{P}), y \in \max(\mathbf{P})\}$. Further, for positive integer $h \geq 2$, let $\mathcal{P}_h$ consist of all posets of height at most h that have planar cover graphs.

For posets with planar cover graphs, we can make the following reduction.

Lemma 2.4.5. *Let h and c be positive integers. If $\mathrm{Crit}^*(\mathbf{P})$ can be partitioned into c reversible sets, for every $\mathbf{P} \in \mathcal{P}_h$, then $\dim(\mathbf{P}) \leq c$ for every $\mathbf{P} \in \mathcal{P}_h$.*

Proof. Let $\mathbf{P} = (X, P)$ be a poset in $\mathcal{P}_h$. Form a poset $\mathbf{Q} \in \mathcal{P}_h$ from $\mathbf{P}$ by adding new elements, all of which will be either minimal elements in $\mathbf{Q}$ or maximal elements of $\mathbf{Q}$. Also, each new minimal element will be covered by a single element of $\mathbf{P}$ and each new maximal element will cover a single element of $\mathbf{P}$, as such:

(1) For each maximal element x of $\mathbf{P}$, add a new minimal point x' which is covered by x in $\mathbf{Q}$.

(2) For each minimal element x of $\mathbf{P}$, add a new point x'' which covers x in $\mathbf{Q}$.

(3) For each element x of $\mathbf{P}$ which is neither maximal nor minimal, add a minimal point x' covered by x in $\mathbf{Q}$ and a maximal point x'' which covers x in $\mathbf{Q}$.

It is easy to see that if $\mathcal{R}$ is a family of linear extensions of $\mathbf{Q}$ that reverses $\mathrm{Crit}^*(\mathbf{Q})$, then restricting the extensions in $\mathcal{R}$ to the elements of $\mathbf{P}$ yields a family of linear extensions of P reversing $\mathrm{Inc}(\mathbf{P})$. $\square$

2.5 The main theorem

The rest of the chapter is devoted to the proof of the following theorem.

Theorem 2.5.1. *For every $h \geq 1$, there exists a least positive integer c_h so that if $\mathbf{P}$ is a poset of height h and the cover graph of $\mathbf{P}$ is planar, then $\dim(\mathbf{P}) \leq c_h$.*

As noted previously, the case $h = 1$ is trivial, and $c_1 = 2$. The case $h = 2$ is very non-trivial, but here we know from Theorem 2.3.1 that $c_2 = 4$. So for the remainder of the proof, we assume that $h \geq 3$. Here, the existence of c_h is not at all clear. However, we will use the remainder of this chapter to show that c_h exists. In particular, the bulk of the chapter is devoted to the proof of an upper bound on c_h, which culminates with Theorem 2.12.7. A discussion of the best-known lower bound on c_h will take place in Section 2.13.

To accomplish our goal of providing an upper bound on c_h, we consider an arbitrary poset $\mathbf{P}$ having a planar cover graph and height h. We then show that the set $\mathrm{Crit}^*(\mathbf{P})$ of incomparable min-max pairs can be partitioned into a small number of reversible sets, where small means bounded as a function of h. To this end, we first handle a special case—although as we will see, this case is actually the heart of the problem. In Theorem 2.11.1 we provide an upper bound for this problem in the special case. We then return to the general case in Section 2.12.

Special Case. There is an $a_0 \in \min(\mathbf{P})$ such that $a_0 < b$ in $\mathbf{P}$ for all $b \in \max(\mathbf{P})$.

Consider a drawing without edge crossings of the cover graph of $\mathbf{P}$ in the plane with the vertex a_0 on the infinite face. From here on, we will refer to the cover graph of $\mathbf{P}$ simply as $\mathbf{G}$.

We consider the edges of $\mathbf{G}$ oriented from x to y when $x < y$ in $\mathbf{P}$. In discussions to follow, we will talk about oriented paths in $\mathbf{G}$. These are sequences $x_0 < x_1 < x_2 < \cdots < x_r$ where x_i is covered by x_{i+1} in $\mathbf{P}$ for each $i = 1, 2, \ldots, r-1$. However, we will also discuss cycles and walks in the general sense, i.e., without any concern for the orientation on the edges.

For convenience, we let A denote the set $\min(\mathbf{P}) - a_0$ and we let $B = \max(\mathbf{P})$. Let T be an oriented tree so that:

(1) T is a subgraph of $\mathbf{G}$;

(2) a_0 is the root of T;

(3) all other vertices in T are on paths oriented away from a_0; and

(4) the elements of B are leaves of T (although perhaps there are leaves of T that are not in B).

Using clockwise orientation to establish precedence, we perform a depth first search of T and this results in a linear order on the vertices of T with the root a_0 as the least element. If an element x is less than another element y in this linear order then we write $x <_T y$. We suggest how the tree T might appear in Figure 2.8.

As a second example, we return to Figure 2.7 and relabel the point which was previously a one to be a minimal element a_0 which is less than each maximal element. We show the resulting figure in Figure 2.9.

Now we have a poset $\mathbf{P}$ satisfying the properties we are assuming in this special case, and we have a suitable drawing with the vertex a_0 on the infinite face. It follows in this example that the oriented tree T is just a star, and the resulting linear order is $a_0 <_T b_1 <_T b_2 <_T \cdots <_T b_8$.

If u and v are vertices of the tree T, we let $\mathrm{T}(u, v)$ denote the unique path in T from u to v. When u is the root a_0 we will write $\mathrm{T}(v)$ instead of $\mathrm{T}(a_0, v)$. Also, let

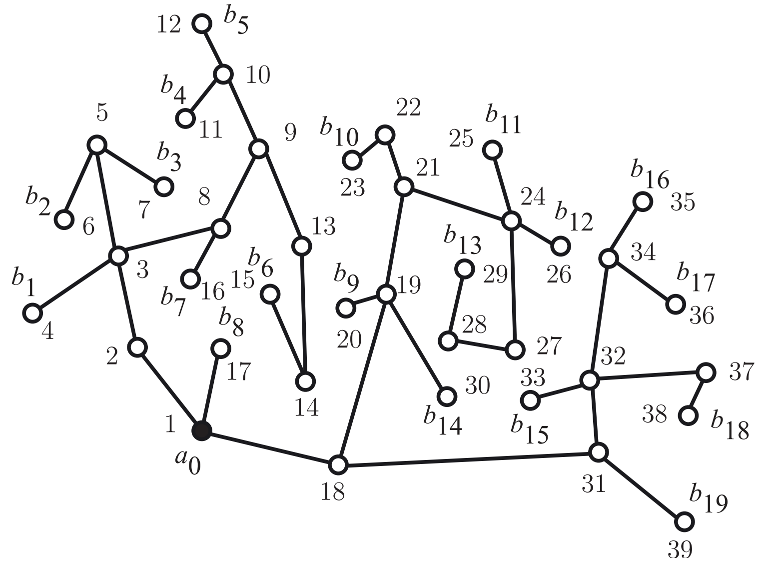


Figure 2.8: The oriented tree T

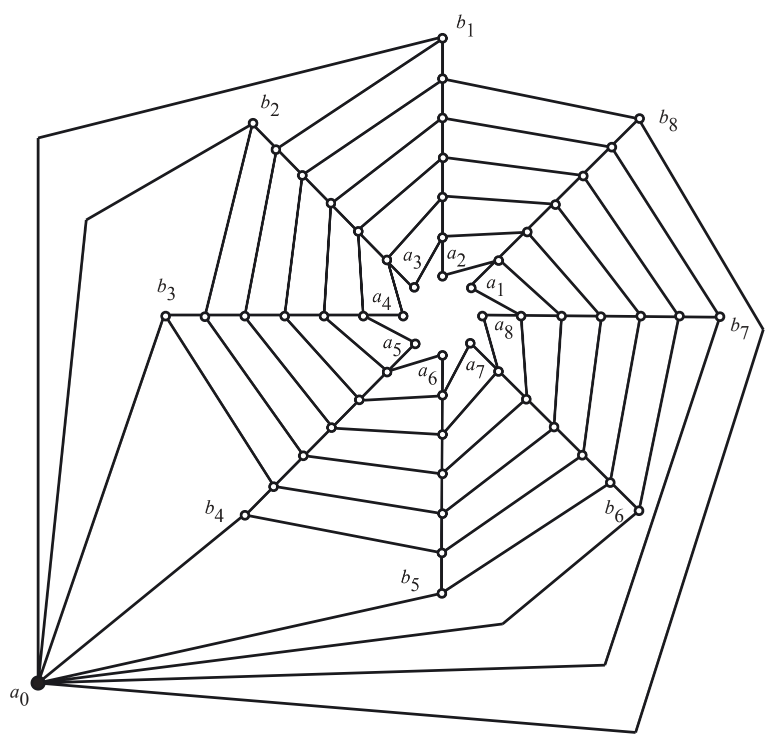


Figure 2.9: A poset with a planar cover graph

T'(v) be T(v) $- v$ and let T'(u, v) be T(u, v) $- \{u, v\}$ when a_0 is neither u nor v. For example, T'(b_1) in Figure 2.8 is the path consisting of the vertices labeled 1, 2, and 3 and T'(b_2, b_7) is the path with vertices labeled 5, 3, and 8. If W is a walk in $\mathbf{G}$ then we let $|W|$ denote the number of vertices in W. Thus $|\,\mathrm{T}(u, v)|$ is the number of vertices on T(u, v).

Now let $a \in A$. Set $\mathrm{Spec}(a) = \{u \in T : a < u$ in $\mathbf{P}$ and $a \| v$ for all $v \in \mathrm{T'}(u)\}$. We say the elements of Spec(a) are the *special points of* a.

Proposition 2.5.2. *If $a \in A$, $b \in B$ and $a < b$ in $\mathbf{P}$, then there is some $s \in \mathrm{Spec}(a)$ so that $s \in \mathrm{T}(b)$.*

We say that a maximal element $b \in B$ is *left-safe for a* if $a \| b$ and $b <_T s$ for every $s \in \mathrm{Spec}(a)$. Similarly, we say that a maximal element b is *right-safe for a* if $a \| b$ and $s <_T b$ for every $s \in \mathrm{Spec}(a)$.

Proposition 2.5.3. *The following two subsets of* $\mathrm{Crit}^*(\mathbf{P})$ *are reversible:*

(1) $\{(a, b) \in \mathrm{Crit}^*(\mathbf{P}) : b$ *is left-safe for* $a\}$.

(2) $\{(a, b) \in \mathrm{Crit}^*(\mathbf{P}) : b$ *is right-safe for* $a\}$.

Proof. Suppose the proposition fails for the first set. Choose an integer $k \geq 2$ and an alternating cycle $\{(a_i, b_i) : 1 \leq i \leq k\}$ with b_i left-safe for a_i for each $i = 1, 2, \ldots, k$. For each i, let w_i denote the least element of Spec(a_i) in the linear order on T. Since $a_i \leq b_{i+1}$ in $\mathbf{P}$, Proposition 2.5.2 guarantees a point $s_i \in \mathrm{Spec}(a_i)$ with $a_i < s_i \leq b_{i+1}$ in $\mathbf{P}$ and s_i on T(b_{i+1}). But this implies $w_i \leq_T s_i \leq_T b_{i+1} <_T w_{i+1}$, and the inequality $w_i <_T w_{i+1}$ cannot hold cyclically. $\square$

So for the remainder of the proof, we consider only critical pairs (a, b) in $\mathrm{Crit}^*(\mathbf{P})$ for which there exist points $v, w \in \mathrm{Spec}(a)$ with $v <_T b <_T w$. We call these pairs *dangerous*. In what follows, we will categorize the dangerous critical pairs by providing for each such pair a *signature*, denoted $\Sigma(a,b)$. This signature records information about the critical pair. The reader can think of $\Sigma(a,b)$ as a vector of parameters. We do not require that these vectors have a common length, nor do we require that the i^{th} coordinate of every vector represent the same parameter. However, we do require the following:

(1) the number of parameters in $\Sigma(a,b)$ is bounded as a function of h, and

(2) the number of distinct values that can be taken by any given coordinate in $\Sigma(a,b)$ is bounded as a function of h.

As a consequence of (1) and (2), the number of distinct signatures is bounded as a function of h. Our goal will be to show that any set of critical pairs with identical signatures can be reversed in the same linear extension of $\mathbf{P}$. If we achieve this, then we have will have proven Theorem 2.5.1.

One of the parameters in $\Sigma(a,b)$ records information about a subset of $\mathrm{Spec}(a)$, which we call the unimodal sequence for a. In general, the size of $\mathrm{Spec}(a)$ can be arbitarily large. So it is important that we are able to bound the size of the unimodal sequence by a function of h. This sequence is also used to determine other parameters in $\Sigma(a,b)$, some of which are defined by curves in the plane that intersect at elements of the unimodal sequence. In particular, since (a,b) is dangerous, we are able to identify a path in T and two paths in the upset of a whose union bounds a well-defined region in the plane that contains b. This region will play a very important role in the remainder of this chapter. We define it formally in the next section.

2.6 Fixed special points

Fix $v, w \in T - a_0$ such that $v <_T w$. Let $a \in A$ with $v, w \in \mathrm{Spec}(a)$. Define $\mathcal{P}_v(a)$ to be the set of oriented paths from a to v in $\mathbf{G}$ and define $\mathcal{P}_w(a)$ to be the set of oriented paths from a to w in $\mathbf{G}$. For each $P_1 \in \mathcal{P}_v(a)$ and $P_2 \in \mathcal{P}_w(a)$, let

$m_{P_1,P_2}(a)$ be the common point of P_1 and P_2 farthest from a. Notice that we might have $m_{P_1,P_2}(a) \in \{v, w\}$.

Consider the following paths: the subpath of P_1 from $m_{P_1,P_2}(a)$ to v; the subpath of P_2 from $m_{P_1,P_2}(a)$ to w; and $\mathrm{T}(v, w)$. According to their definitions, and because $v, w \in \mathrm{Spec}(a)$, each path is internally disjoint from the others. So, the union of these paths is a Jordan curve, and as such bounds a well-defined region in the plane [52]. Call this region $\mathfrak{R}_{P_1,P_2}(a)$.

Now consider $\mathfrak{R}_{P_i,P_j}(a)$ for every pair of paths $P_i \in \mathcal{P}_v(a)$ and $P_j \in \mathcal{P}_w(a)$. These regions are partially ordered by inclusion. Arbitrarily select a minimal element in this partial order, say $\mathfrak{R}_{P_{i'},P_{j'}}(a)$, to be fixed for the remainder of this chapter. From here on we shall refer to this region as $\mathfrak{R}_{v,w}(a)$. We will also refer to $m_{P_{i'},P_{j'}}(a)$ as $m_{v,w}(a)$, to the subpath of $P_{i'}$ from $m_{v,w}(a)$ to v as $P_v(a)$, and to the subpath of $P_{j'}$ from $m_{v,w}(a)$ to w as $P_w(a)$. Notice that a is not on $P_v(a)$ or $P_w(a)$ unless $a = m_{v,w}(a)$. To avoid confusion, we shall refer to the boundary of $\mathfrak{R}_{v,w}(a)$ as $\partial(\mathfrak{R}_{v,w}(a))$, to the part of the plane bounded by $\partial(\mathfrak{R}_{v,w}(a))$ as $\mathrm{In}(\mathfrak{R}_{v,w}(a))$, and to the complementary unbounded part of the plane as $\mathrm{Ex}(\mathfrak{R}_{v,w}(a))$.

The following fact is an immediate consequence of the definitions above and the fact that we have chosen an embedding of $\mathbf{G}$ with a_0 on the infinite face.

Fact 2.6.1. Let $a \in A$ and $v, w \in \mathrm{Spec}(a)$. Then a_0 is embedded in $\mathrm{Ex}(\mathfrak{R}_{v,w}(a))$.

The next facts are easily verified using the minimality of $\mathfrak{R}_{v,w}(a)$ and the nonexistence of directed cycles in $\mathbf{G}$.

Fact 2.6.2. If $k \geq 1$, $(x_0, x_1, \ldots, x_k)$ is a directed path with $x_0, x_k \in P_v(a)$, and $\{x_1, x_2, \ldots, x_{k-1}\} \cap \partial(\mathfrak{R}_{v,w}(a)) = \varnothing$, then $x_i \in \mathrm{Ex}(\mathfrak{R}_{v,w}(a))$ for $i \in [k-1]$. The analogous statement holds for $P_w(a)$.

Fact 2.6.3. If $k \geq 1$, $(x_0, x_1, \ldots, x_k)$ is a directed path with $x_0 \in P_v(a)$ and $x_k \in P_w(a)$, and $\{x_1, x_2, \ldots, x_{k-1}\} \cap \partial(\mathfrak{R}_{v,w}(a)) = \varnothing$, then $x_i \in \mathrm{Ex}(\mathfrak{R}_{v,w}(a))$ for $i \in [k-1]$.

Further, if $k = 1$, then the edge (x_0, x_k) satisfies $(x_0, x_k) \cap \mathrm{In}(\mathfrak{R}_{v,w}(a)) = \varnothing$. The analogous statement holds for $x_0 \in P_w(a)$ and $x_k \in P_v(a)$.

Combining Facts 2.6.2 and 2.6.3, we obtain the following.

Fact 2.6.4. If $\{x_0, x_k\} \subseteq P_v(a) \cup P_w(a)$ and there is a directed path $\mathcal{P}$ from x_0 to x_k, then $\mathcal{P} \cap \mathrm{In}(\mathfrak{R}_{v,w}(a)) = \varnothing$.

We can now establish an important lemma.

Lemma 2.6.5. *If $a_1, a_2 \in A$ with $\{v, w\} \subseteq \mathrm{Spec}(a_1) \cap \mathrm{Spec}(a_2)$, then $P_v(a_1) \cap P_v(a_2)$ is a path.*

Proof. Assume not. Then there exist points x, y such that (1) $P_v(a_1)$ and $P_v(a_2)$ coincide from y to v, (2) x is the greatest element of $P_v(a_1) \cap P_v(a_2)$ such that $x < y$ in $\mathbf{P}$, and (3) x is not covered by y. Let $\mathcal{P}_1$ be the subpath of $P_v(a_1)$ from x to y and let $\mathcal{P}_2$ be the subpath of $P_v(a_2)$ from x to y. Thus $\mathcal{P}_1 \cup \mathcal{P}_2$ is a Jordan curve. Denote the interior of the bounded region defined by this curve by $\mathcal{R}$. See Figure 2.10 for an example.

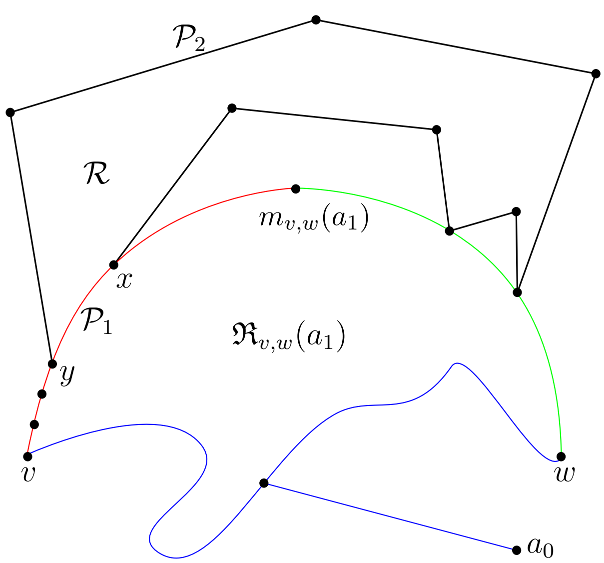


Figure 2.10

By Fact 2.6.4, $\mathcal{P}_1 \cap \mathrm{In}(\mathfrak{R}_{v,w}(a_2)) = \varnothing$ and $\mathcal{P}_2 \cap \mathrm{In}(\mathfrak{R}_{v,w}(a_1)) = \varnothing$. So $\mathcal{R}$ is contained in both $\mathrm{Ex}(\mathfrak{R}_{v,w}(a_1))$ and $\mathrm{Ex}(\mathfrak{R}_{v,w}(a_2))$. Therefore, $\mathrm{In}(\mathfrak{R}_{v,w}(a_1))$ and $\mathrm{In}(\mathfrak{R}_{v,w}(a_2))$

contain points on opposite sides of the common subpath of $P_v(a_1) \cap P_v(a_2)$ from y to v, and so they contain points on opposite sides of $\mathrm{T}(v)$ as well. But then, either $\mathrm{In}(\mathfrak{R}_{v,w}(a_1))$ or $\mathrm{In}(\mathfrak{R}_{v,w}(a_2))$ must contain a_0, contrary to Fact 2.6.1. $\square$

The following corollary is an immediate consequence of Lemma 2.6.5.

Corollary 2.6.6. *Let $a_1, a_2, \ldots, a_k \in A$ with $\{v, w\} \subseteq \bigcap_{i=1}^k \mathrm{Spec}(a_i)$. Then $\bigcup_{i=1}^k P_v(a_i)$ and $\bigcup_{i=1}^k P_w(a_i)$ are subtrees of $\boldsymbol{G}$.*

This leads to the following lemma.

Lemma 2.6.7. *Let $a_1, a_2 \in A$ with $\{v, w\} \subseteq \mathrm{Spec}(a_1) \cap \mathrm{Spec}(a_2)$. If $m_{v,w}(a_2) \notin \mathrm{In}(\mathfrak{R}_{v,w}(a_1))$, then $\mathfrak{R}_{v,w}(a_2)$ and $\mathfrak{R}_{v,w}(a_1)$ are inclusion-wise comparable.*

Proof. Suppose not and let $m_{v,w}(a_2) \in \partial(\mathfrak{R}_{v,w}(a_1))$. Without loss of generality, we may assume $m_{v,w}(a_2) \in P_w(a_1)$. Then, by Corollary 2.6.6, $P_w(a_2)$ is a subpath of $P_w(a_1)$. If $(P_v(a_2) \cap P_w(a_1)) - m_{v,w}(a_2) \neq \varnothing$, then $\mathbf{G}$ has a directed cycle, a contradiction. Let $x \in P_v(a_2)$ be such that $m_{v,w}(a_2) < x$ is a cover relation in $\mathbf{P}$. If $x \in \mathrm{In}(\mathfrak{R}_{v,w}(a_1))$, then, since $P_v(a_1) \cup P_v(a_2)$ is a tree by Corollary 2.6.6, we find that $\mathfrak{R}_{v,w}(a_1)$ properly contains $\mathfrak{R}_{v,w}(a_2)$. Otherwise we find $\mathfrak{R}_{v,w}(a_2)$ properly contains $\mathfrak{R}_{v,w}(a_1)$.

Now let $m_{v,w}(a_2) \in \mathrm{Ex}(\mathfrak{R}_{v,w}(a_1))$. Since we have assumed the regions to be inclusion-wise incomparable, either $P_v(a_2)$ or $P_w(a_2)$ has a point in $\mathrm{In}(\mathfrak{R}_{v,w}(a_1))$. Assume first that $P_v(a_2)$ has such a point and let x be the point closest to $m_{v,w}(a_2)$ in $P_v(a_2) \cap \partial(\mathfrak{R}_{v,w}(a_1))$. If $x \in T - \{v, w\}$, then we have contradicted the fact that $v, w \in \mathrm{Spec}(a_2)$. If $x \in P_v(a_1)$, then, since $P_v(a_1) \cup P_v(a_2)$ induces a tree by Corollary 2.6.6, there cannot be a point of $P_v(a_2)$ on the interior of $\mathfrak{R}_{v,w}(a_1)$. So $x \in P_w(a_2) - m_{v,w}(a_1)$.

Let $\{x_1, x_2, \ldots, x_k\} = P_v(a_2) \cap P_w(a_1)$ where $x = x_1$. Since all points belong to a directed path, we may assume that $x_i < x_{i+1}$ in $\mathbf{P}$ for all $i \in [k-1]$. In fact, these points must appear consecutively in both $P_v(a_2)$ and $P_w(a_1)$ as else we would contradict the minimality of either $\mathfrak{R}_{v,w}(a_1)$ or $\mathfrak{R}_{v,w}(a_2)$. Consider the point $x_{k+1} \in P_v(a_2)$ such that $x_k < x_{k+1}$ is a cover relation of $\mathbf{P}$. If $x_{k+1} \in \text{Ex}(\mathfrak{R}_{v,w}(a_1))$, then $P_v(a_2)$ cannot have a point in $\text{In}(\mathfrak{R}_{v,w}(a_1))$. So we may assume $x_{k+1} \in \text{In}(\mathfrak{R}_{v,w}(a_1))$. The next point in $P_v(a_2)$ that intersects $\partial(\mathfrak{R}_{v,w}(a_1))$ must be in $P_v(a_1)$. This contradicts the minimality of $\mathfrak{R}_{v,w}(a_1)$.

The case for $P_w(a_2)$ follows analogously. $\square$

2.6.1 Standard position

In this section, we define a parameter that we will use to classify dangerous critical pairs. This parameter will be defined on the minimal elements of the critical pairs in such a way that any two elements with the same parameter-value determine inclusion-wise comparable regions. To this end, we say that two inclusion-wise incomparable regions $\mathfrak{R}_{v,w}(a_1)$ and $\mathfrak{R}_{v,w}(a_2)$ are in *standard position with* $a_1 < a_2$ *in order* $\mathcal{L}_{v,w}$, or simply $a_1 <_{\mathcal{L}_{v,w}} a_2$, if the following is true: $m_{v,w}(a_i)$ is in the interior of $\mathfrak{R}_{v,w}(a_{3-i})$ for $i \in \{1,2\}$, $P_w(a_1) \cap P_v(a_2) \neq \varnothing$, $P_v(a_1) \cap P_w(a_2) = \varnothing$, and the embedding of $\mathbf{P}$ restricted to $\mathfrak{R}_{v,w}(a_1)$ and $\mathfrak{R}_{v,w}(a_2)$ has faces $F'_{1,2}$ and $F''_{1,2}$ bounded only by $\{P_v(a_1), P_w(a_1), P_v(a_2)\}$ and $\{P_w(a_1), P_v(a_2), P_w(a_2)\}$, respectively. Figure 2.11 illustrates a case in which $a_1 <_{\mathcal{L}_{v,w}} a_2$.

Lemma 2.6.8. *Let* $a_1, a_2 \in A$ *with* $\{v,w\} \subseteq \text{Spec}(a_1) \cap \text{Spec}(a_2)$. *If* $\mathfrak{R}_{v,w}(a_1)$ *and* $\mathfrak{R}_{v,w}(a_2)$ *are inclusion-wise incomparable, then either* $a_1 <_{\mathcal{L}_{v,w}} a_2$ *or* $a_2 <_{\mathcal{L}_{v,w}} a_1$.

Proof. By Lemma 2.6.7 it must be the case that $m_{v,w}(a_i) \in \text{In}(\mathfrak{R}_{v,w}(a_{3-i}))$ for $i \in \{1,2\}$. Now consider the embedding of $\partial(\mathfrak{R}_{v,w}(a_1))$ with $m_{v,w}(a_2)$ somewhere in the interior. Since $P_v(a_1) \cup P_v(a_2)$ and $P_w(a_1) \cup P_w(a_2)$ are trees by Corollary 2.6.6, it must be the case that at least one of $P_v(a_1) \cap P_w(a_2)$ and $P_w(a_1) \cap P_v(a_2)$ is nonempty, as else the regions are inclusion-wise comparable.

Assume first that $P_v(a_2) \cap P_w(a_1) = \{x_1, x_2, \ldots, x_k\} \neq \varnothing$. We may assume $x_i <$

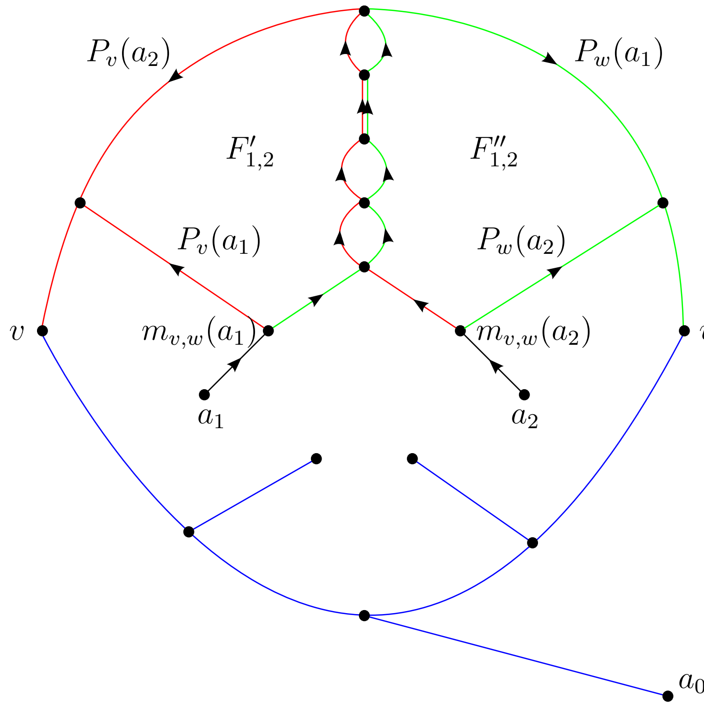


Figure 2.11: Inclusion-wise incomparable regions $\mathfrak{R}_{v,w}(a_1)$ and $\mathfrak{R}_{v,w}(a_2)$

x_{i+1} in $\mathbf{P}$ for all $i \in [k-1]$ since these points are all on a directed path. Notice that these points must occur in the same order in both paths, as else $\mathbf{G}$ has a directed cycle, but they do not have to occur consecutively in either path. However, any $y \in P_v(a_2)$ with $x_i < y < x_{i+1}$ for some $i \in [k-1]$ must be embedded in $\mathrm{Ex}(\mathfrak{R}_{v,w}(a_1))$, as else we use Fact 2.6.4 to contradict the minimality of $\mathfrak{R}_{v,w}(a_1)$.

Let $x_{k+1} \in P_v(a_2)$ be such that $x_k < x_{k+1}$ is a cover relation of $\mathbf{P}$. If $x_{k+1} \in \mathrm{In}(\mathfrak{R}_{v,w}(a_1))$, then again we use Fact 2.6.4 to contradict the minimality of $\mathfrak{R}_{v,w}(a_1)$. Therefore, the subpath of $P_v(a_2)$ from x_{k+1} to v intersects $\partial(\mathfrak{R}_{v,w}(a_1)) - v$ in only $P_v(a_1)$. Recalling that a_0 is in the infinite face of the embedding, we can find $F'_{1,2}$.

Now consider $P_w(a_2)$. If $P_w(a_2)$ intersects $P_v(a_1)$, then we can use Fact 2.6.4 to contradict the minimality of $\mathfrak{R}_{v,w}(a_1)$, or we find a point other than $m_{v,w}(a_2)$ in the both $P_w(a_2)$ and $P_v(a_2)$, contradicting the choice of $m_{v,w}(a_2)$. Thus $P_w(a_2)$ does not intersect $P_v(a_1)$, and as such we can find $F''_{1,2}$ in the embedding. Therefore, the regions are in standard position with $a_1 <_{\mathcal{L}_{v,w}} a_2$.

Assume second that $P_w(a_2) \cap P_v(a_1) \neq \varnothing$. An analogous argument to the one above implies that the regions are in standard position with $a_2 <_{\mathcal{L}_{v,w}} a_1$. $\square$

The next lemma states that $\mathcal{L}_{v,w}$ is transitive.

Lemma 2.6.9. *Let $a_1, a_2, a_3 \in A$ with $\{v, w\} \subseteq \mathrm{Spec}(a_1) \cap \mathrm{Spec}(a_2) \cap \mathrm{Spec}(a_3)$. If $a_1 <_{\mathcal{L}_{v,w}} a_2$ and $a_2 <_{\mathcal{L}_{v,w}} a_3$, then $a_1 <_{\mathcal{L}_{v,w}} a_3$.*

Proof. We must show that $\mathfrak{R}_{v,w}(a_1)$ and $\mathfrak{R}_{v,w}(a_3)$ are inclusion-wise incomparable. Suppose not. Without loss of generality, we may assume that $m_{v,w}(a_3) \in \mathrm{Ex}(\mathfrak{R}_{v,w}(a_1))$. Since $a_2 <_{\mathcal{L}_{v,w}} a_3$, we know $m_{v,w}(a_3) \in \mathrm{In}(\mathfrak{R}_{v,w}(a_2))$. Since $a_1 <_{\mathcal{L}_{v,w}} a_2$, we find that $m_{v,w}(a_3)$ must be embedded in $F'_{1,2}$. The fact that $a_2 <_{\mathcal{L}_{v,w}} a_3$ implies that $P_w(a_3)$ cannot intersect $P_v(a_2)$. Notice that $P_w(a_3)$ cannot intersect $P_w(a_1)$ on the boundary of $F'_{1,2}$; if they do, Corollary 2.6.6 implies that $P_w(a_3)$ and $P_w(a_1)$ coincide to w, in which case $P_w(a_3)$ intersects $P_v(a_2)$. So $P_w(a_3)$ must exit $F'_{1,2}$ through $P_v(a_1) - m_{v,w}(a_1)$ and proceed into $\mathrm{In}(\mathfrak{R}_{v,w}(a_1))$. But this contradicts the minimality of $\mathfrak{R}_{v,w}(a_1)$, using Fact 2.6.4. Therefore $\mathfrak{R}_{v,w}(a_1)$ and $\mathfrak{R}_{v,w}(a_3)$ are inclusion-wise incomparable. From this argument we can also deduce that $m_{v,w}(a_3) \in \mathrm{In}(\mathfrak{R}_{v,w}(a_1)) \cap \mathrm{In}(\mathfrak{R}_{v,w}(a_2))$.

By Lemma 2.6.8, we know that $\mathfrak{R}_{v,w}(a_1)$ and $\mathfrak{R}_{v,w}(a_3)$ are in standard position, and as such $a_1 <_{\mathcal{L}_{v,w}} a_3$ or $a_3 <_{\mathcal{L}_{v,w}} a_1$. Suppose, by way of contradiction, that $a_3 <_{\mathcal{L}_{v,w}} a_1$. Consider the embedding of $\mathfrak{R}_{v,w}(a_1)$ and $\mathfrak{R}_{v,w}(a_2)$, recalling that $m_{v,w}(a_3)$ is embedded in the interior of each. Since $a_2 <_{\mathcal{L}_{v,w}} a_3$, we see that $P_w(a_2)$ and $P_v(a_3)$ intersect in a nonempty set $\{x_1, x_2, \ldots, x_k\}$. We may assume $x_i < x_{i+1}$ for all $i \in [k-1]$, since all points lie on a directed path. Let $x_{k+1} \in P_v(a_3)$ be such that $x_k < x_{k+1}$ is a cover relation in $\mathbf{P}$. If the edge $x_k x_{k+1}$ intersects $\mathrm{Ex}(\mathfrak{R}_{v,w}(a_2))$, then $P_v(a_3)$ exits $F''_{1,2}$ through $P_w(a_1)$ or $P_v(a_2)$. But $P_v(a_3)$ cannot intersect either of these paths on the boundary of $F''_{1,2}$, as each implies that $P_v(a_3) \cap P_w(a_1) \neq \varnothing$, contrary to the assumption that $a_3 <_{\mathcal{L}_{v,w}} a_1$ (we are using Corollary 2.6.6 in the case that $P_v(a_3)$ intersects $P_v(a_2)$). Therefore the edge $x_k x_{k+1}$ is in $\mathrm{In}(\mathfrak{R}_{v,w}(a_1)) \cap \mathrm{In}(\mathfrak{R}_{v,w}(a_2))$. In particular, $P_v(a_3)$ stays in the interior of each region until it intersects $P_v(a_1)$. But this contradicts the minimality of $\mathfrak{R}_{v,w}(a_2)$, using the following paths: the subpath of $P_w(a_2)$ from x_k to w; the subpath of $P_v(a_3)$ from x_k to v; $\mathrm{T}(v,w)$. $\square$

For each $a \in A$, with fixed special points $v, w \in \mathrm{Spec}(a)$, define $\pi^{\mathrm{sp}}_{v,w}(a)$ to be the length of a longest sequence $a = a_1, a_2, \ldots, a_l$ such that $a_i <_{\mathcal{L}_{v,w}} a_{i+1}$ for each $i \in [l-1]$. Notice that, *a priori*, there is no bound on $\pi^{\mathrm{sp}}_{v,w}(a)$. However, we can bound this parameter from above in terms of h. Let $t \in \mathbb{N}$ be sufficiently large such that any partition of the two-element subsets of $[t]$ into h^2 classes results in a three-element subset $\{i, j, k\}$ and class α so that $\{i, j\}$, $\{i, k\}$, and $\{j, k\}$ are all in α. Denote the minimum such t by $R_2(3, h^2)$, which exists by Ramsey's theorem [41].

Proposition 2.6.10. *Let $a \in A$ with $v, w \in \mathrm{Spec}(a)$. Then $\pi^{sp}_{v,w}(a) < R_2(3, h^2)$.*

Proof. Let $a = a_1, a_2, \ldots, a_l$ be the sequence defining $\pi^{\mathrm{sp}}_{v,w}(a)$. Let a_i and a_j be any two elements of this sequence with $i < j$. By Lemma 2.6.9 we know that $a_i <_{\mathcal{L}_{v,w}} a_j$. Thus $P_w(a_i) \cap P_v(a_j) = X \neq \varnothing$. Let $x_{i,j}$ be the minimal element of X in $\mathbf{P}$. Color each pair $\{i, j\}$ with the color (y, z), where y is the length of the subpath of $P_w(a_i)$ from $m_{v,w}(a_i)$ to $x_{i,j}$, and z is the length of the subpath of $P_v(a_j)$ from $m_{v,w}(a_j)$ to $x_{i,j}$. Clearly there are at most h^2 colors used in this scheme, as $y, z \leq h$. If $l \geq R_2(3, h^2)$, then there is some color, say α, and three indices $i < j < k$, such that the pairs $\{i, j\}$, $\{i, k\}$, and $\{j, k\}$ receive color α. Consider $x_{i,k}$ (notice that $x_{i,k} \neq m_{v,w}(a_j)$ by Lemma 2.6.7). Since $\{i, k\}$ and $\{i, j\}$ are colored α, we must have $x_{i,k} = x_{i,j}$. Since $\{i, k\}$ and $\{j, k\}$ are colored α, we must have $x_{i,k} = x_{j,k}$. But then $x_{i,k}$ is in both $P_v(a_j)$ and $P_w(a_j)$, contradicting the choice of $m_{v,w}(a_2)$. $\square$

Let $a_1, a_2 \in A$ such that $\mathfrak{R}_{v,w}(a_1)$ and $\mathfrak{R}_{v,w}(a_2)$ are inclusion-wise incomparable. We have seen that either $a_1 <_{\mathcal{L}_{v,w}} a_2$ or $a_2 <_{\mathcal{L}_{v,w}} a_1$. It follows that $\pi^{\mathrm{sp}}_{v,w}(a_1) \neq \pi^{\mathrm{sp}}_{v,w}(a_2)$. Therefore we have proven the following result, which we will state as a theorem for emphasis.

Theorem 2.6.11. *Fix $v, w \in T$ and $k_1, k_2 \in \mathbb{N}$. Let $a_1, a_2 \in A$ with $\{v, w\} \subseteq \mathrm{Spec}(a_1) \cap \mathrm{Spec}(a_2)$, $\pi^{sp}_{v,w}(a_1) = k_1$, and $\pi^{sp}_{v,w}(a_2) = k_2$. Then $k_1, k_2 < R_2(3, h^2)$, and if $k_1 = k_2$, then $\mathfrak{R}_{v,w}(a_1)$ and $\mathfrak{R}_{v,w}(a_2)$ are inclusion-wise comparable.*

2.7 Fixed regions

In this section, we will only be concerned with critical pairs whose minimal elements determine regions that are identical. Call this region $\mathcal{R}$, and let $v <_T w$ be the special points defining $\mathcal{R}$. Further, since all $(a, b) \in \mathrm{Crit}^*(\mathbf{P})$ with region $\mathcal{R}$ share $m_{v,w}(a)$, we will refer to this point as m.

2.7.1 Interior minimal points

Define $\mathcal{A}_\mathcal{R}$ to be the set of $a \in A$ such that $v, w \in \mathrm{Spec}(a)$, $\mathfrak{R}_{v,w}(a) = \mathcal{R}$, and there exists $b \in B$ with $(a, b) \in \mathrm{Crit}^*(\mathbf{P})$ such that both a and b are embedded in $\mathrm{In}(\mathcal{R})$. For each $a \in \mathcal{A}_\mathcal{R}$, consider the paths in $\mathbf{G}$ oriented from a to m. Define a depth first search tree $\tau(a)$ with root m on the graph induced by these paths, using locally-clockwise preferences and starting the orientation at the first edge in $P_w(a)$ (but clearly traversing each edge in the opposite direction to its orientation). Let $P_m(a)$ be the path in $\tau(a)$ from m to a.

Since $P_m(a_1) \cap P_m(a_2)$ must be a path for any $a_1, a_2 \in \mathcal{A}_\mathcal{R}$ (otherwise we constructed some DFS tree incorrectly), we see that the union of the $P_m(a)$ over all $a \in \mathcal{A}_\mathcal{R}$ is a tree. Call this tree τ_m. Using clockwise orientation to establish precedence, we label the elements of τ_m using another depth first search. This results in a linear order L_m on the vertices of τ_m with the root m as the least element. (Notice that $a_1 <_{L_m} a_2$ in Figure 2.12a even though it appears to the right; clockwise here starts on the right.)

For each $(a, b) \in \mathrm{Crit}^*(\mathbf{P})$ with $a \in \mathcal{A}_\mathcal{R}$, define an *$\mathcal{R}_L$-sequence starting at* (a, b) as a list of critical pairs in $(a, b) = (a_1, b_1), (a_2, b_2), \ldots, (a_l, b_l)$ such that for all $i \in [l-1]$,

(1) $a_i <_{L_m} a_{i+1}$;

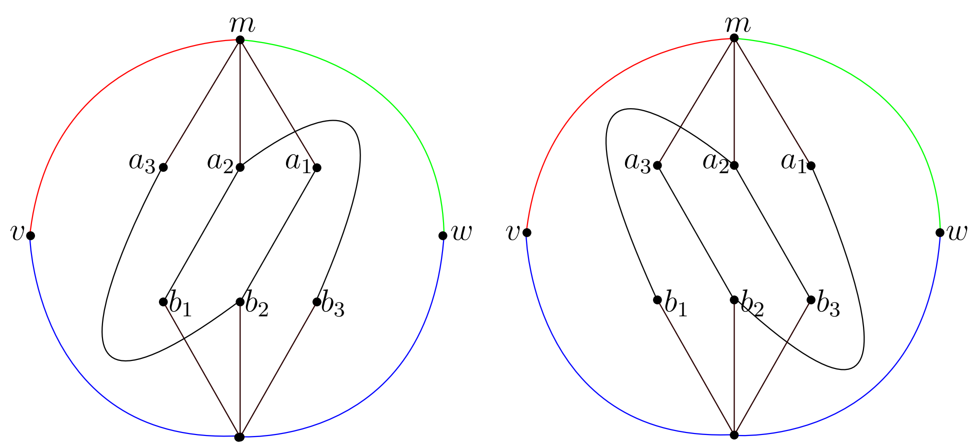
(2) $b_i <_T b_{i+1}$;

(3) $a_i < b_{i+1}$ in $\mathbf{P}$; and

(4) $a_{i+1} < b_i$ in $\mathbf{P}$.

For each $(a, b) \in \mathrm{Crit}^*(\mathbf{P})$ with $a \in \mathcal{A}_{\mathcal{R}}$, define $\pi^{\mathcal{R}\mathrm{L}}(a, b)$ to be the length of the longest $\mathcal{R}_L$-sequence starting at (a, b). The following definitions and lemmas illustrate that this parameter can be bounded from above in terms of h.

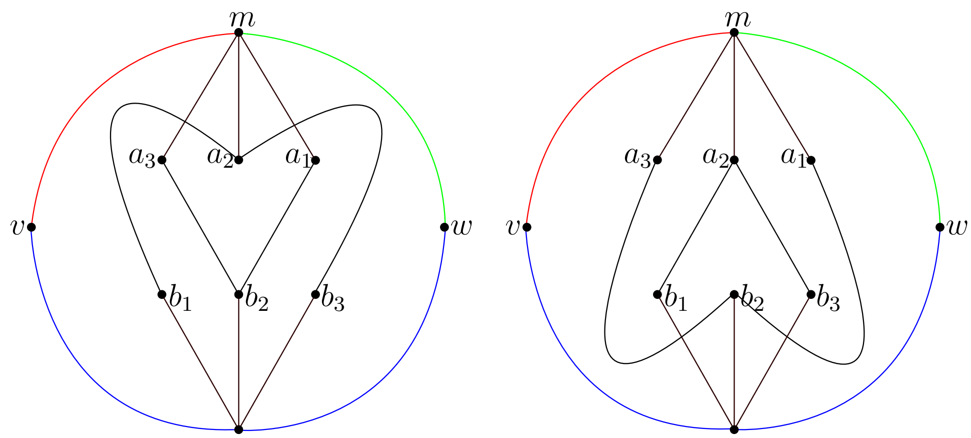
Consider $\mathcal{R}_L$ sequences of length three. In particular, consider the sequences depicted in Figures 2.12 and 2.13. The sequences in Figure 2.12 we refer to as Type 1 and Type 1^D, respectively (the D is for *dual*), and the sequences in Figure 2.13 we refer to as Type 2 and Type 2^D, respectively. We claim that all $\mathcal{R}_L$-sequences of length three are of one of these types.



(a) Type 1 (b) Type 1^D

Figure 2.12

Before we proceed, we need to make the definitions of these length-three sequences more formal. Let (a_1, b_1), (a_2, b_2), (a_3, b_3) be an $\mathcal{R}_L$-sequence starting at (a_1, b_1). The definitions above require that $a_i < b_j$ for some i, j. This comparability is witnessed by an oriented path in $\mathbf{G}$. Fix one such path for each (a_i, b_j) pair and call it $\mathcal{P}(a_i, b_j)$. Then the path-intersections depicted in Figures 2.12 and 2.13 are precisely those that



(a) Type 2 (b) Type 2^D

Figure 2.13

determine the type of sequence, with the lone exception that two paths may intersect if they do not imply comparabilities between $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ not already implied by the depicted paths. For example, if the sequence is Type 1, then $\mathcal{P}(a_2, b_3)$ intersects $P_m(a_2)$, $P_m(a_1)$, and $\mathrm{T}(b_3)$, and does not intersect $P_m(a_3)$, $\mathrm{T}(b_1)$, $\mathrm{T}(b_2)$, $\mathcal{P}(a_2, b_1)$, $\mathcal{P}(a_1, b_2)$, or $\mathcal{P}(a_3, b_2)$. However, $\mathcal{P}(a_1, b_2)$ might intersect $\mathrm{T}(b_3)$, since we have already determined that $a_1 < b_3$ by the intersection of $P_m(a_1)$ and $\mathcal{P}(a_2, b_3)$. (Note that these intersections may not occur as drawn in the Figures. For example, it may be the case that $\mathcal{P}(a_2, b_3)$ intersects T'(b_3) before arriving at b_3.)

If the only paths that $\mathcal{P}(a_i, b_j)$ intersects are $P_m(a_i)$ and $\mathrm{T}(b_j)$, or possibly a path that does not imply any comparabilities between $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ not already implied by the depicted paths, then we say that $\mathcal{P}(a_i, b_j)$ is *direct* or that it goes *directly* from a_i to b_j. For example, $\mathcal{P}(a_1, b_2)$ in a Type 1 sequence is direct, even if it intersects $\mathrm{T}(b_3)$. Lastly, note that $\mathcal{P}(a_i, b_j) \cap \partial(\mathcal{R})$ is empty for all i, j.

Lemma 2.7.1. *Let (a_1, b_1), (a_2, b_2), (a_3, b_3) be an $\mathcal{R}_L$-sequence starting at (a_1, b_1) of length three. This sequence is Type 1, Type 1^D, Type 2, or Type 2^D.*

Proof. As we are only concerned with path-intersections, we may assume that T, $\mathcal{R}$, and τ_m are embedded as they appear in Figures 2.12 and 2.13. By the definition of the $\mathcal{R}_L$-sequence, we have $a_2 < b_3$, $a_3 < b_2$, $a_1 < b_2$, and $a_2 < b_1$. Consider the embedding of $\mathcal{P}(a_2, b_3)$ and let x be the first point of intersection of $\mathcal{P}(a_2, b_3)$ in $\mathrm{T}(b_3)$. We find $\mathcal{P}(a_2, b_3)$ is disjoint from both $P_m(a_3)$ and $\mathrm{T}(b_2)$; otherwise $a_3 < b_3$ or $a_2 < b_2$, respectively. If $\mathcal{P}(a_2, b_3)$ intersects $\mathrm{T}(b_1)$, with y maximal in $\mathbf{P}$ such that $y \in \mathcal{P}(a_2, b_3) \cap \mathrm{T}(b_1)$, then b_2 is in the interior of the Jordan curve formed by: the subpath of $\mathcal{P}(a_2, b_3)$ from y to x; the subpath of $\mathrm{T}(b_1)$ ending at x; and the subpath of $\mathrm{T}(b_3)$ ending at y. Therefore we cannot embed $\mathcal{P}(a_3, b_2)$ without forcing $a_3 < b_3$, as each point on the boundary of this curve is less than b_3 in $\mathbf{P}$. Thus, $\mathcal{P}(a_2, b_3)$ intersects only $P_m(a_1)$ or goes directly from b_2 to a_3.

Assume first that $\mathcal{P}(a_2, b_3)$ intersects only $P_m(a_1)$, and consider the embedding of $\mathcal{P}(a_1, b_2)$. We see $\mathcal{P}(a_1, b_2)$ is disjoint from $\mathcal{P}(a_2, b_3)$; otherwise $a_2 < b_2$. If $\mathcal{P}(a_1, b_2)$ intersects $P_m(a_3)$, then, by an argument similar to the one above using $\mathcal{P}(a_1, b_2)$, $P_m(a_1)$, and $P_m(a_3)$ to bound a region containing a_2, we cannot embed $\mathcal{P}(a_2, b_1)$ without forcing $a_2 < b_2$. Clearly $\mathcal{P}(a_1, b_2)$ does not intersect $\mathrm{T}(b_1)$, so $\mathcal{P}(a_1, b_2)$ must go directly from a_1 to b_2.

Consider the embedding of $\mathcal{P}(a_2, b_1)$. It cannot intersect $\mathcal{P}(a_1, b_2)$ or $\mathrm{T}(b_2)$ without making $a_2 < b_2$. It cannot intersect $P_m(a_1)$ without making $a_1 < b_1$. So it either (1) is direct, or (2) intersects only $P_m(a_3)$. In both cases, it is easily verified that $\mathcal{P}(a_3, b_2)$ has only one option for its embedding: if (1) holds, it intersects only $\mathrm{T}(b_1)$, and if (2) holds, it goes directly from a_3 to b_2. The first case is Type 1, and the second case is Type 2.

Now assume that $\mathcal{P}(a_2, b_3)$ goes directly from b_2 to a_3. By a similar analysis, we find that $\mathcal{P}(a_1, b_2)$ intersects only $\mathrm{T}(b_3)$. Then $\mathcal{P}(a_2, b_1)$ either is direct or intersects only $P_m(a_3)$. In the first case, $\mathcal{P}(a_3, b_2)$ intersects only $\mathrm{T}(b_1)$ and in the second case $\mathcal{P}(a_3, b_2)$ is direct. The first case is Type 2^D and the second is Type 1^D. $\square$

The following Corollary is an immediate consequence of Lemma 2.7.1, only needing a bit of inspection to verify. It states that the relation defined for consecutive pairs in an $\mathcal{R}_L$-sequence is transitive.

Corollary 2.7.2. *Let* (a_1, b_1), (a_2, b_2), (a_3, b_3) *be an* $\mathcal{R}_L$*-sequence starting at* (a_1, b_1) *of length three. Then* (a_1, b_1), (a_3, b_3) *is an* $\mathcal{R}_L$*-sequence starting at* (a_1, b_1) *of length two.*

Let $t \in \mathbb{N}$ be sufficiently large such that any partition of the three-element subsets of $[t]$ into four classes results in an $(h+1)$-element subset whose three-element subsets are all in the same class. Denote the minimum such t by $R_3(h+1, 4)$, which exists by Ramsey's theorem.

Lemma 2.7.3. *For each* $(a, b) \in \mathrm{Crit}^*(\mathbf{P})$ *with* $a \in \mathcal{A}_{\mathcal{R}}$ *we have the following upper bound:* $\pi^{\mathcal{R}L}(a, b) < R_3(h+1, 4)$.

Proof. Consider an $\mathcal{R}_L$-sequence of length at least $R_3(h+1, 4)$. Notice that any subsequence of length three is also an $\mathcal{R}_L$-sequence, and as such is of Type 1, Type 1^D, Type 2, or Type 2^D by Lemma 2.7.1. Color all subsequences of length three according to their type; give color 1 for Type 1, give color 2 for Type 1^D, give color 3 for Type 2, and give color 4 for Type 2^D. By Ramsey's Theorem, there is a monochromatic set of triples, say $\mathcal{M}$, of size $h+1$.

Notice that, by the definition of an $\mathcal{R}_L$-sequence, the triples in $\mathcal{M}$ induce an $\mathcal{R}_L$-sequence of length $|\mathcal{M}|$. Name this sequence (a_1, b_1), (a_2, b_2), $\ldots$, (a_{h+1}, b_{h+1}). As shorthand, we will refer to any subsequence of length three by the indices of its elements (e.g. (a_1, b_1), (a_2, b_2), (a_3, b_3) will be referred to as $\{1, 2, 3\}$). The remainder of the proof focuses on the sequences in the following subsets of $\mathcal{M}$:

$$\mathcal{M}' = \big\{\{1,2,3\}, \{1,3,4\}, \ldots, \{1, h, h+1\}\big\}, \text{ and}$$

$$\mathcal{M}'' = \big\{\{h-1, h, h+1\}, \{h-2, h-1, h+1\}, \ldots, \{1, 2, h+1\}\big\}.$$

Assume first that all triples in $\mathcal{M}$ have color 1. Consider $\mathcal{P}(a_3, b_2)$ from $\{1, 2, 3\}$, remembering that it intersects $\mathrm{T}(b_1)$. Call this point of intersection x_1. Now consider

$\mathcal{P}(a_4, b_3)$ from $\{1,3,4\}$. It also intersects $\mathrm{T}(b_1)$, say at x_2. But x_2 must be strictly less than x_1 in $\mathbf{P}$ as else $a_3 < b_3$. Repeating this argument for any $\{1,i,i+1\}, \{1,i+1,i+2\} \in \mathcal{M}'$ yields a sequence $x_1 < x_2 < \cdots < x_{h-1}$. But $x_i < b_1$ in $\mathbf{P}$ for all $i \in [h-1]$, since $x_i \leq x_1 < b_2$. Further, $x_i > u$ in $\mathbf{P}$ for all $i \in [h-1]$, since otherwise $a_i \leq u < b_j$ for all $j \in [h+1]$, which would imply $a_i < b_i$. Thus $\mathrm{T}(b_1)$ has length at least $h+1$, contradicting the height of $\mathbf{P}$. See Figure 2.14.

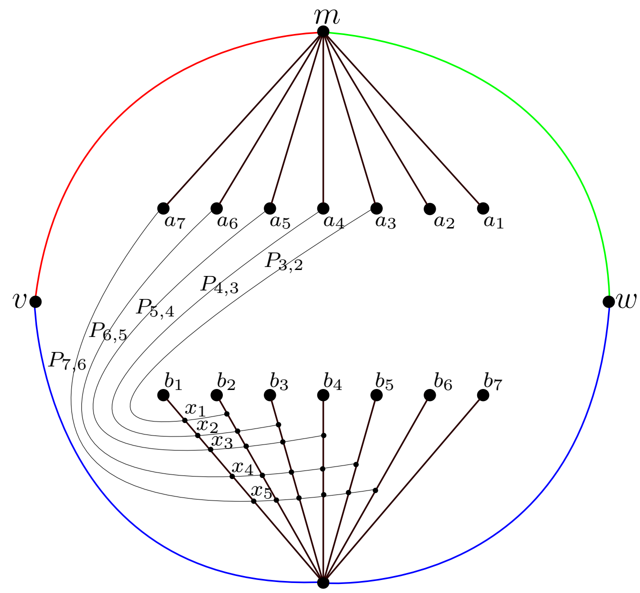


Figure 2.14: The Type 1 case for $h = 6$. Here we have written $P_{i,j}$ to mean $\mathcal{P}(a_i, b_j)$.

Second, assume that all triples in $\mathcal{M}$ have color 2. If we use $\mathcal{M}''$ instead of $\mathcal{M}'$, path $\mathcal{P}(a_i, b_{i+1})$ from $\{i, i+1, h+1\}$ instead of path $\mathcal{P}(a_{i+1}, b_i)$ from $\{1, i, i+1\}$, and $\mathrm{T}(b_{h+1})$ instead of $\mathrm{T}(b_1)$, and if we apply the analogous analysis to the previous case, we see that $\mathrm{T}(b_{h+1})$ has length at least $h+1$, a contradiction.

Third, assume that all triples in $\mathcal{M}$ have color 3. If we use $\mathcal{M}'$, path $\mathcal{P}(a_i, b_{i+1})$ from $\{1, i, i+1\}$ instead of path $\mathcal{P}(a_{i+1}, b_i)$ from $\{1, i, i+1\}$, and $P_m(a_1)$ instead of $\mathrm{T}(b_1)$, and if we apply the analogous analysis to the first case, we see that $P_m(a_1)$ has length at least $h+1$, a contradiction.

Last, assume that all triples in $\mathcal{M}$ have color 4. If we use $\mathcal{M}''$ instead of $\mathcal{M}'$, path $\mathcal{P}(a_{i+1}, b_i)$ from $\{i, i+1, h+1\}$ instead of path $\mathcal{P}(a_{i+1}, b_i)$ from $\{1, i, i+1\}$, and

$P_m(a_{h+1})$ instead of $\mathrm{T}(b_1)$, and if we apply the analogous analysis to the first case, we see that $P_m(a_{h+1})$ has length at least $h+1$, a contradiction. □

Before we state the main theorem of this section, we need a similar definition to the one above. For each $(a,b) \in \mathrm{Crit}^*(\mathbf{P})$ with $a \in \mathcal{A}_{\mathcal{R}}$, define an $\mathcal{R}_R$*-sequence starting at* (a,b) as a list of critical pairs in $(a,b) = (a_1,b_1),(a_2,b_2),\ldots,(a_r,b_r)$ such that for all $i \in [r-1]$,

(1) $a_i >_{L_m} a_{i+1}$;

(2) $b_i >_T b_{i+1}$;

(3) $a_i < b_{i+1}$ in $\mathbf{P}$; and

(4) $a_{i+1} < b_i$ in $\mathbf{P}$.

For each $(a,b) \in \mathrm{Crit}^*(\mathbf{P})$ with $a \in \mathcal{A}_{\mathcal{R}}$, define $\pi^{\mathcal{R}\mathrm{R}}(a,b)$ to be the length of the longest $\mathcal{R}_R$-sequence starting at (a,b). The analogs of Lemmas 2.7.1 and 2.7.3 imply that we get the same bound on $\mathcal{R}_R$-sequences as we have on $\mathcal{R}_L$-sequences; $\pi^{\mathcal{R}\mathrm{R}}(a,b) < R_3(h+1,4)$ for all $a \in \mathcal{A}_{\mathcal{R}}$. We can now prove the main theorem of this section.

Theorem 2.7.4. *Fix $k_1, k_2 \in \mathbb{N}$ with each less than $R_3(h+1,4)$. The following set of critical pairs is reversible:*

$$\mathcal{S} = \{(a,b) \in \mathrm{Crit}^*(\mathbf{P}) \mid a \in \mathcal{A}_{\mathcal{R}},\ \pi^{\mathcal{R}L}(a,b) = k_1,\ \pi^{\mathcal{R}R}(a,b) = k_2\}.$$

Proof. Suppose not and let $\mathcal{C} = \{(a_1,b_1),(a_2,b_2),\ldots,(a_r,b_r)\}$ be a strict alternating cycle of length r with $a_i < b_{i+1}$ cyclically. As before, fix a directed path witnessing the fact that $a_i < b_j$ and call it $\mathcal{P}(a_i,b_j)$. Because the alternating cycle is strict, the paths $\{\mathcal{P}(a_i,b_{i+1})\}_{i=1}^r$ are disjoint. Furthermore, if $\mathcal{P}(a_i,b_{i+1})$ intersects $P_m(a_j)$ or $\mathrm{T}(b_{j'})$, then the strictness forces $j = i$ and $j' = i+1$; that is, $\mathcal{P}(a_i,b_{i+1})$ goes directly from a_i to b_{i+1}. Thus, when embedded (and "straightened"), these paths look like a matching of size r between the minimal elements and the maximal elements.

We wish to assign labels to the coordinates of pairs in the alternating cycle. We do so with function f in the following way: $\mathrm{f}(b_i) = n$ if b_i appears n^{th} in the linear order on T when the linear order is restricted to the maximal elements in $\mathcal{C}$, and $\mathrm{f}(a_i) = n$ if a_i appears n^{th} in the reverse order of τ_m when τ_m is restricted to the minimal elements in $\mathcal{C}$. For example, in Figure 2.14, $\mathrm{f}(b_i) = i$ and $\mathrm{f}(a_i) = 8 - i$ (both sets of labels start on the left). Notice that $\mathrm{f}(a_i) = \mathrm{f}(b_j)$ if and only if $j = i + 1$.

We call $(a_i, b_i) \in \mathcal{C}$ *left* if $\mathrm{f}(a_i) > \mathrm{f}(b_i)$ and *right* if $\mathrm{f}(a_i) < \mathrm{f}(b_i)$. Notice that, by the previous remark, $\mathrm{f}(a_i) \neq \mathrm{f}(b_i)$. Therefore, each pair is either left or right. If a_i, a_j, b_k, and b_l appear as coordinates of critical pairs in $\mathcal{C}$, we say that (a_i, b_k) is *left-crossing* of (a_j, b_l) when $\mathrm{f}(a_i) < \mathrm{f}(a_j)$, $\mathrm{f}(b_k) > \mathrm{f}(b_l)$, $a_i < b_l$ in $\mathbf{P}$, and $a_j < b_k$ in $\mathbf{P}$. In the same case we would say (a_j, b_l) is *right-crossing* of (a_i, b_k).

Consider two distinct critical pairs (a_i, b_i) and (a_j, b_j) for which (a_i, b_i) is right, (a_j, b_j) is left, and $\mathrm{f}(a_i) = \mathrm{f}(b_j)$. (For example, we can pick i such that $\mathrm{f}(a_i) = 1$ and then $j = i+1$.) Notice that $\mathrm{f}(a_j) = \mathrm{f}(b_i)$ if and only if $r = 2$. But then it is clear that $\pi^{\mathcal{R}\mathrm{L}}(a_j, b_j) \geq \pi^{\mathcal{R}\mathrm{L}}(a_i, b_i) + 1$, contradicting the definition of $\mathcal{S}$. So we may assume that $\mathrm{f}(a_j) \neq \mathrm{f}(b_i)$. This presents two cases; either $\mathrm{f}(a_i) = \mathrm{f}(b_j) < \mathrm{f}(a_j) < \mathrm{f}(b_i)$, represented in Figure 2.15a, or $\mathrm{f}(a_i) = \mathrm{f}(b_j) < \mathrm{f}(b_i) < \mathrm{f}(a_j)$, represented in Figure 2.15b.

Assume first that $\mathrm{f}(a_j) < \mathrm{f}(b_i)$. Let b_k be such that $\mathrm{f}(b_k) = \mathrm{f}(a_j)$. Notice that the critical pair (a_i, b_k), while not in the alternating cycle, is nevertheless left-crossing of (a_j, b_j). (The pair (a_i, b_k) is in $\mathcal{A}_{\mathcal{R}}$ since $\mathcal{C}$ is a strict alternating cycle, and so a_i and b_k are incomparable.) We will prove that $\pi^{\mathcal{R}\mathrm{L}}(a_i, b_i) \neq \pi^{\mathcal{R}\mathrm{L}}(a_j, b_j)$ by showing that any critical pair left-crossing of (a_i, b_i) is also left-crossing of (a_i, b_k). In particular, we will show $\pi^{\mathcal{R}\mathrm{L}}(a_j, b_j) \geq \pi^{\mathcal{R}\mathrm{L}}(a_i, b_i) + 1$, a contradiction.

Let (a_l, b_l) be a critical pair left-crossing of (a_i, b_i) and assume (a_l, b_l) is not left-crossing of (a_i, b_k). (If no such pair exists, then our claim is verified since we would

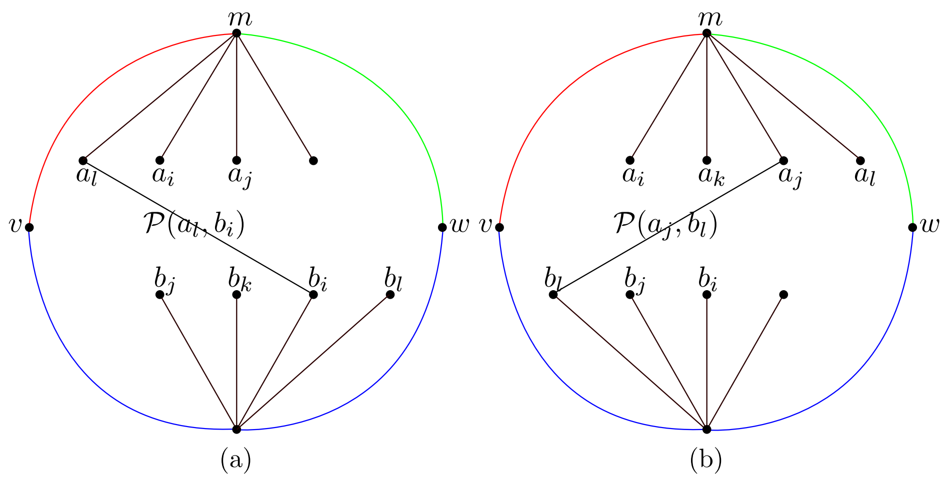


Figure 2.15: Two cases in the proof of Theorem 2.7.4

find $\pi^{\mathcal{R}\mathrm{L}}(a_i, b_i) = 0$ and $\pi^{\mathcal{R}\mathrm{L}}(a_j, b_j) \geq 1$.) Consider the path $\mathcal{P}(a_l, b_i)$. It must be embedded directly since it cannot intersect $P_m(a_i)$ or $\mathrm{T}(b_k)$; in the former $a_i < b_i$, and in the latter (a_l, b_l) is left-crossing of (a_i, b_k), contrary to our assumption. Moreover, $\mathcal{P}(a_l, b_i)$ cannot intersect $P_m(a_j)$, as otherwise we contradict the strictness of $\mathcal{C}$. See Figure 2.15a. Now consider $\mathcal{P}(a_j, b_k)$. It cannot intersect $P_m(a_l)$ or $\mathcal{P}(a_l, b_i)$; if so, (a_l, b_l) is left-crossing of (a_i, b_k), contrary to our assumption. Furthermore, $\mathcal{P}(a_j, b_k)$ cannot intersect $\mathrm{T}(b_i)$; otherwise, $a_j < b_i$, contrary to the strictness of $\mathcal{C}$. But then, planarity implies that there is no way to embed this path, a contradiction.

Now assume $\mathrm{f}(a_j) > \mathrm{f}(b_i)$. Let $a_k \in \mathcal{A}_{\mathcal{R}}$ such that $\mathrm{f}(a_k) = \mathrm{f}(b_i)$. Notice that the critical pair (a_k, b_j) is right-crossing of (a_i, b_i). So, we will show that $\pi^{\mathcal{R}\mathrm{R}}(a_i, b_i) \neq \pi^{\mathcal{R}\mathrm{R}}(a_j, b_j)$ by showing that any critical pair right-crossing of (a_j, b_j) is also right-crossing of (a_k, b_j). In particular, we will show $\pi^{\mathcal{R}\mathrm{R}}(a_i, b_i) \geq \pi^{\mathcal{R}\mathrm{R}}(a_j, b_j) + 1$, a contradiction.

Let (a_l, b_l) be a critical pair right-crossing of (a_j, b_j) and assume (a_l, b_l) is not right-crossing of (a_k, b_j). At this point, we can use an argument analogous to that of the previous case to show first that $\mathcal{P}(a_j, b_l)$ goes directly from a_j to b_l, as in Figure 2.15b, and second that there is no valid embedding of $\mathcal{P}(a_k, b_i)$. $\square$

2.7.2 Exterior and boundary minimal elements

Define $\mathcal{A}'_{\mathcal{R}}$ to be all $a \in A$ such that $v, w \in \operatorname{Spec}(a)$, $\mathfrak{R}_{v,w}(a) = \mathcal{R}$, and there exists $b \in B$ with $(a, b) \in \operatorname{Crit}^*(\mathbf{P})$ such that $b \in \operatorname{In}(\mathcal{R})$ and $a \notin \operatorname{In}(\mathcal{R})$. The following proposition states that all such critical pairs can be reversed with just one linear extension.

Proposition 2.7.5. *The following set of critical pairs is reversible:*

$$\mathcal{S} = \{(a, b) \in \operatorname{Crit}^*(\mathbf{P}) \mid a \in \mathcal{A}'_{\mathcal{R}}\}.$$

Proof. Let $a \in \mathcal{A}'_{\mathcal{R}}$ with a embedded on $\partial(\mathcal{R})$. Since a is a minimal element of $\mathbf{P}$ it must be the case that $a = m$ and that $\mathcal{A}'_{\mathcal{R}} = \{a\}$. Therefore, since all critical pairs in $\mathcal{S}$ have the same first-coordinate, $\mathcal{S}$ does not contain an alternating cycle.

We may now assume that all $a \in \mathcal{A}'_{\mathcal{R}}$ are embedded in $\operatorname{Ex}(\mathcal{R})$. Suppose that $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ is an alternating cycle of length k with $a_i < b_{i+1}$ cyclically and with each critical pair in $\mathcal{S}$. Let $\mathcal{P}(a_1, b_2)$ be any path in $\mathbf{G}$ directed from a_1 to b_2. Since $a_1 \in \operatorname{Ex}(\mathcal{R})$ and $b_2 \in \operatorname{In}(\mathcal{R})$, we see that $\mathcal{P}(a_1, b_2)$ intersects $\partial(\mathcal{R})$. Since $v, w \in \operatorname{Spec}(a_i)$ for all $i \in [k]$, we find that $\mathcal{P}(a_1, b_2) \cap (\mathrm{T}(v) \cup \mathrm{T}(u)) = \varnothing$. Thus $\mathcal{P}(a_1, b_2)$ intersects $P_v(a_2) \cup P_w(a_2)$, contradicting the fact that a_2 and b_2 are incomparable. $\square$

2.8 Partitioning the critical pairs

The purpose of this section is to identify the regions that we will use to classify the critical pairs in $\operatorname{Crit}^*(\mathbf{P})$. Equivalently, we will identify the special points that will be used to define each region.

2.8.1 Unimodal sequences

For all $a \in A$ let $l_0(a) = \min\{|\mathrm{T}(s)| : s \in \operatorname{Spec}(a)\}$ and define $L_0(a) = \{s \in \operatorname{Spec}(a) : |\mathrm{T}(s)| = l_0(a)\}$. Let $\lambda_0(a)$ be the least element of $L_0(a)$ in the linear order on T.

For $i \geq 1$ we define $l_i(a) = \min\{|\,\mathrm{T}(s)| : s \in \mathrm{Spec}(a), s <_T \lambda_{i-1}(a)\}$, we define $L_i(a) = \{s \in \mathrm{Spec}(a) : |\,\mathrm{T}(s)| = l_i(a), s <_T \lambda_{i-1}(a)\}$, and we let $\lambda_i(a)$ be the least element of $L_i(a)$ in the linear order on T. Notice that $i \leq h$ since $|\,\mathrm{T}(s)| \leq h$ for all $s \in \mathrm{Spec}(a)$, so we have at most h such sets.

The next group of definitions is analogous to the previous set. Set $r_0(a) = l_0(a)$ and $R_0(a) = L_0(a)$. Define $\omega_0(a)$ to be the greatest element of $R_0(a)$ in the linear order on T. Then for $i \geq 1$ we define $r_i(a) = \min\{|\,\mathrm{T}(s)| : s \in \mathrm{Spec}(a), s >_T \omega_{i-1}(a)\}$, we define $R_i(a) = \{s \in \mathrm{Spec}(a) : |\,\mathrm{T}(s)| = r_i(a), s >_T \omega_{i-1}(a)\}$, and we let $\omega_i(a)$ be the greatest element of $R_i(a)$ in the linear order on T.

Let $\Lambda(a)$ be the least element of $\mathrm{Spec}(a)$ in the linear order on T and let $\Omega(a)$ be the greatest element of $\mathrm{Spec}(a)$ in the same order. The following fact is an immediate consequence of the preceding definitions.

Fact 2.8.1. Let m be the greatest integer such that $L_m(a)$ is nonempty and let n be the greatest integer such that $R_n(a)$ is nonempty. Then $\Lambda(a) = \lambda_m(a)$ and $\Omega(a) = \omega_n(a)$.

For $a \in A$ let m and n be integers such that $\Lambda(a) = \lambda_m(a)$ and $\Omega(a) = \omega_n(a)$. Define the *unimodal sequence for a* to be

$$(l_m(a), l_{m-1}(a), \ldots, l_1(a), l_0(a) = r_0(a), r_1(a), \ldots, r_{n-1}(a), r_n(a))\,.$$

In Section 2.10, we will be considering alternating cycles amongst critical pairs whose minimal elements share the same unimodal sequence. Therefore we need to bound the number of such sequences as a function of h.

Proposition 2.8.2. *There are at most 2^{2h} distinct unimodal sequences over all $a \in A$.*

Proof. We can think of a unimodal sequence as a binary sequence of length $2h$. The first h positions represent the sets $L_i(a)$ and the second h positions represent the sets $R_i(a)$, for $1 \leq i \leq h$. We record a 1 if the represented set is nonempty and a 0 otherwise. Then two unimodal sequences are the same if and only if their corresponding binary sequences are the same. $\square$

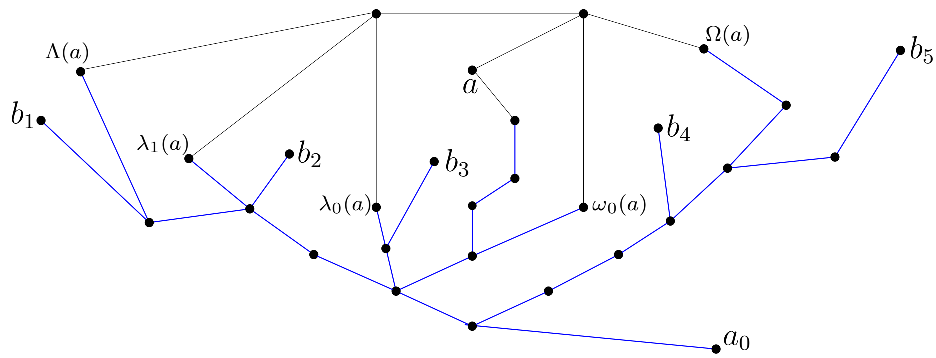


Figure 2.16: Critical pair (a, b_1) is left-safe, (a, b_2) is left-dangerous, (a, b_3) is center-dangerous, (a, b_4) is right-dangerous, and (a, b_5) is right-safe. The unimodal sequence for a is $(7, 6, 5, 5, 8)$.

2.8.2 Classification

Let $(a, b) \in \text{Crit}^*(P)$. We call (a, b) *center-dangerous* if $\lambda_0(a) <_T b <_T \omega_0(a)$. If $\Lambda(a) < b <_T \lambda_0(a)$, then we call (a, b) *left-dangerous*. If $\omega_0(a) <_T b <_T \Omega(a)$, then we call (a, b) *right-dangerous*. See Figure 2.16.

Proposition 2.8.3. *All* $(a, b) \in \text{Crit}^*(\mathbf{P})$ *are left-safe, left-dangerous, center-dangerous, right-dangerous, or right-safe.*

Proof. By definition, if $b <_T \Lambda(a)$ then (a, b) is left-safe and if $b >_T \Omega(a)$ then (a, b) is right-safe. Since b is incomparable to all elements of $\text{Spec}(a)$, the statement follows from Fact 2.8.1 and the definitions above. $\square$

2.9 Center-dangerous critical pairs

The goal of this section is to define a signature for center-dangerous critical pairs so that no set of center-dangerous critical pairs whose elements have identical signatures contains an alternating cycle. To this end, for $a \in A$ define its *center-region* as $\mathfrak{R}_{\lambda_0(a), \omega_0(a)}(a)$. Because this notation is quite cumbersome, we will instead refer to

$\lambda_0(a)$ as $\beta(a)$, or just β if the context is clear, and we will refer to $\omega_0(a)$ as $\gamma(a)$, or just γ if the context is clear. Furthermore, since the special points that determine the center-region for a have been specified, we will write $\mathfrak{R}(a)$ instead of $\mathfrak{R}_{\beta,\gamma}(a)$ and $m(a)$ instead of $m_{\beta,\gamma}(a)$.

Lemma 2.9.1. *Let (a, b) and (a', b') be center-dangerous critical pairs whose center-regions are defined by β, γ, and β', γ', respectively. If $|\,\mathrm{T}(\beta)| = |\,\mathrm{T}(\beta')|$ and $\pi^{sp}_{\beta,\gamma}(a) = \pi^{sp}_{\beta',\gamma'}(a')$, then either (1) $\mathfrak{R}(a)$ and $\mathfrak{R}(a')$ are inclusion-wise comparable or have disjoint interiors, or (2) $\beta <_T \beta' <_T \gamma <_T \gamma'$, $m(a) \in \mathrm{Ex}(\mathfrak{R}(a'))$, $m(a') \in \mathrm{Ex}(\mathfrak{R}(a))$, $P_\gamma(a) \cap P_{\beta'}(a') \neq \varnothing$, and $P_\beta(a) \cap P_{\beta'}(a') = P_\gamma(a) \cap P_{\gamma'}(a') = P_\beta(a) \cap P_{\gamma'}(a') = \varnothing$.*

Proof. Note that $x \| y$ for each pair $x, y \in \{\beta, \gamma, \beta', \gamma'\}$ with $x \neq y$, since $|\,\mathrm{T}(\beta)| = |\,\mathrm{T}(\gamma)| = |\,\mathrm{T}(\beta')| = |\,\mathrm{T}(\gamma')|$. Also $(P_\beta(a) \cup P_\gamma(a)) - \{\beta, \gamma\}$ cannot intersect $\mathrm{T}(\beta') \cup \mathrm{T}(\gamma')$ without contradicting the fact that $\beta, \gamma \in L_0(a)$, and $(P_{\beta'}(a') \cup P_{\gamma'}(a')) - \{\beta', \gamma'\}$ cannot intersect $\mathrm{T}(\beta) \cup \mathrm{T}(\gamma)$ without contradicting the fact that $\beta', \gamma' \in L_0(a')$.

Assume we are not in condition (1). If $\beta = \beta'$ and $\gamma = \gamma'$, then Theorem 2.6.11 implies that $\mathfrak{R}(a)$ and $\mathfrak{R}(a')$ are inclusion-wise comparable. So, without loss of generality, we may assume that $\beta <_T \beta'$.

Now assume $\gamma \leq_T \beta'$. If $\mathrm{In}(\mathfrak{R}(a)) \cap \mathrm{In}(\mathfrak{R}(a')) \neq \varnothing$, then we may assume, without loss of generality, that $P_\beta(a)$ or $P_\gamma(a)$ has an edge in $\mathrm{In}(\mathfrak{R}_{\beta',\gamma'}(a'))$. In the former case, we contradict the minimality of β' in the linear order on T. If $P_\beta(a)$ does not have an edge in $\mathrm{In}(\mathfrak{R}_{\beta',\gamma'}(a'))$, then $m(a) \notin \mathrm{In}(\mathfrak{R}(a'))$, and we find that a subpath of $P_\gamma(a)$ can be used to contradict Fact 2.6.4 for $\mathfrak{R}(a')$.

Next assume that $\gamma' \leq_T \gamma$. Note that $(P_{\beta'}(a') \cup P_{\gamma'}(a')) \cap P_\beta(a) = \varnothing$, as else we contradict the minimality in β' in the linear order on T. If $m(a') \in \mathrm{Ex}(\mathfrak{R}(a))$, then we contradict the maximality of γ' in the linear order on T unless $\gamma' = \gamma$. In this case, Corollary 2.6.6 implies that $P_\gamma(a) \cap P_{\gamma'}(a')$ is a path. Let x be maximal in $\mathbf{P}$ such that $x \in P_{\beta'}(a') \cap P_\gamma(a)$. Then the union of the subpath of $P_{\beta'}(a')$ from x to β', the subpath of $P_\gamma(a)$ from x to γ, and the path in T from β' to γ is a Jordan curve. The region inside this curve contradicts the minimality of $\mathfrak{R}(a')$. If $m(a') \in \partial(\mathfrak{R}(a))$, then $m(a') = x$ in the argument above and we conclude that $\mathfrak{R}(a)$ and $\mathfrak{R}(a')$ are inclusion-wise comparable. So we may assume $m(a') \in \mathrm{In}(\mathfrak{R}(a))$. If $P_{\beta'}(a')$ or $P_{\gamma'}(a')$ has an edge in $\mathrm{Ex}(\mathfrak{R}(a))$, then we contradict Fact 2.6.4 for $\mathfrak{R}(a')$. So the center-regions must be inclusion-wise comparable.

Therefore, we may assume that $\beta <_T \beta' <_T \gamma <_T \gamma'$. As a consequence, $\beta' \in \mathrm{In}(\mathfrak{R}(a))$ and $\gamma \in \mathrm{In}(\mathfrak{R}(a'))$. If $m(a') \notin \mathrm{Ex}(\mathfrak{R}(a))$, then consider $P_{\gamma'}(a')$. It must intersect either $P_\beta(a)$ or $P_\gamma(a)$. In the former case, we contradict the minimality of β' in the linear order on T, and in the latter case, we contradict the maximality of γ in the linear order on T. If $m(a') \in \mathrm{Ex}(\mathfrak{R}(a))$, then consider $P_{\beta'}(a')$. It must intersect either $P_\beta(a)$ or $P_\gamma(a)$. In the former case, we contradict the minimality of β' in the linear order on T. In the latter case, we get condition (2), as otherwise we contradict the minimality of β' or the maximality of γ in the linear order on T. $\square$

In the next lemma, we write m_j instead of $m(a_j)$ to simplify the notation.

Lemma 2.9.2. *Let $\{(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\}$ be a set of center-dangerous critical pairs which form a strict alternating cycle of length k with $a_i < b_{i+1}$ cyclically. Suppose the center-regions for these pairs are defined by β_i and γ_i for each $i \in [k]$. Further, suppose $|\mathrm{T}(\beta_i)| = |\mathrm{T}(\beta_j)|$, $\pi^{sp}_{\beta_i,\gamma_i}(a_i) = \pi^{sp}_{\beta_j,\gamma_j}(a_j)$, $|P_{\beta_i}(a_i)| = |P_{\beta_j}(a_j)|$, and $|P_{\gamma_i}(a_i)| = |P_{\gamma_j}(a_j)|$ for each $i, j \in [k]$. Then $\mathfrak{R}_{\beta_i,\gamma_i}(a_i)$ and $\mathfrak{R}_{\beta_j,\gamma_j}(a_j)$ are identical for all $i, j \in [k]$.*

Proof. Choose an index i such that $\mathfrak{R}(a_i)$ is inclusion-wise minimal amongst all center-regions $\mathfrak{R}(a_j)$ for $j \in [k]$. Recall that $b_i \in \mathrm{In}(\mathfrak{R}(a_i))$. We wish to locate a_{i-1}. By way of contradiction, assume that $a_{i-1} \notin \mathrm{In}(\mathfrak{R}(a_i))$. Let $\mathcal{P}(a_{i-1}, b_i)$ be any directed path in $\mathbf{G}$ from a_{i-1} to b_i. Since $\partial(\mathfrak{R}(a_i))$ is a Jordan curve, it must be the case that $\mathcal{P}(a_{i-1}, b_i)$ intersects $\partial(\mathfrak{R}(a_i))$. But $\mathcal{P}(a_{i-1}, b_i) \cap (P_{\beta_i}(a_i) \cup P_{\gamma_i}(a_i))$ must be empty; otherwise, we contradict the fact that a_i and b_i are incomparable. Also $\mathcal{P}(a_{i-1}, b_i) \cap (\mathrm{T}(\beta_i) \cup \mathrm{T}(\gamma_i))$

must be empty; otherwise, we contradict the fact that $\beta_i, \gamma_i \in \mathrm{Spec}(a_i)$. Therefore $a_{i-1} \in \mathrm{In}(\mathfrak{R}(a_i))$.

Suppose $m_{i-1} \in \mathrm{Ex}(\mathfrak{R}(a_i))$. Let $\mathcal{P}(a_{i-1}, m_{i-1})$ be any path in $\mathbf{G}$ from a_{i-1} to m_{i-1}. Clearly $\mathcal{P}(a_{i-1}, m_{i-1}) \cap \mathrm{T'}(\beta_i, \gamma_i)$ is empty, so $\mathcal{P}(a_{i-1}, m_{i-1})$ intersects $P_{\beta_i}(a_i)$ or $P_{\gamma_i}(a_i)$. In the former case, $a_{i-1} < \beta_i$ in $\mathbf{P}$, so $\beta_{i-1} \leq_T \beta_i$. If $\beta_{i-1} <_T \beta_i$ then we contradict the minimality of β_i in the linear order on T, as $a_i < m_i < \beta_{i-1}$ in $\mathbf{P}$. So $\beta_{i-1} = \beta_i$. So we must be in condition (1) of Lemma 2.9.1, and thus $\gamma_i \leq_T \gamma_{i-1}$. However, if $\gamma_i <_T \gamma_{i-1}$ then we contradict the maximality of γ_i in the linear order on T, as $a_i < m_{i-1} < \gamma_{i-1}$ in $\mathbf{P}$. Thus $\gamma_i = \gamma_{i-1}$. Let x be maximal in $\mathbf{P}$ such that $x \in \mathcal{P}(a_{i-1}, m_{i-1}) \cap P_{\beta_i}(a_i)$. Then the region bounded by the following paths contradicts the minimality of $\mathfrak{R}(a_{i-1})$: the subpath of $P_{\beta_i}(a_i)$ from x to β_i; the subpath of $\mathcal{P}(a_{i-1}, m_{i-1})$ from x to m_{i-1}; the path $P_{\gamma_{i-1}}(a_{i-1})$; and the path from β_i to γ_i in T. The latter case, in which $\mathcal{P}(a_{i-1}, m_{i-1})$ intersects $P_{\gamma_i}(a_i)$, follows analogously.

Now suppose $m_{i-1} \in \mathrm{In}(\mathfrak{R}(a_i))$. Condition (1) of Lemma 2.9.1 implies $\mathfrak{R}(a_{i-1})$ is properly contained in $\mathfrak{R}(a_i)$, contradicting the fact that $\mathfrak{R}(a_i)$ is inclusion-wise minimal over the center-regions associated with the critical pairs in the alternating cycle. Therefore $m_{i-1} \in \partial(\mathfrak{R}(a_i))$. Clearly $\mathcal{P}(a_{i-1}, m_{i-1}) \cap (\mathrm{T}(\beta_i) \cup \mathrm{T}(\gamma_i))$ is empty, so m_{i-1} is on $P_{\beta_i}(a_i)$ or $P_{\gamma_i}(a_i)$. If $m_{i-1} \neq m_i$, then, by Corollary 2.6.6, either $|P_{\beta_{i-1}}(a_{i-1})| \neq |P_{\beta_i}(a_i)|$ or $|P_{\gamma_{i-1}}(a_{i-1})| \neq |P_{\gamma_i}(a_i)|$, respectively. Both are contradictions. So $m_{i-1} = m_i$, in which case $\mathfrak{R}(a_{i-1})$ and $\mathfrak{R}(a_i)$ are identical. Applying this argument cyclically yields the desired result. $\square$

Define the parameter $\pi^{\mathrm{in}}(a, b)$ to be 1 if a is embedded in $\mathrm{In}(\mathfrak{R}_{\beta,\gamma}(a))$ and to be 0 otherwise. For every center-dangerous critical pair (a, b), define $\Sigma(a, b)$ as the vector with the following coordinates:

- $|\,\mathrm{T}(\beta)| = |\,\mathrm{T}(\gamma)|$,

- $|P_\alpha(a)|$,
- $|P_\beta(a)|$,
- $\pi^{\mathrm{sp}}_{\beta,\gamma}(a)$,
- $\pi^{\mathcal{R}\mathrm{L}}(a,b)$,
- $\pi^{\mathcal{R}\mathrm{R}}(a,b)$, and
- $\pi^{\mathrm{in}}(a,b)$,

where we have substituted $\mathcal{R}$ for $\mathfrak{R}_{\beta,\gamma}(a)$ to simplify notation. We can now prove the main result of this section.

Theorem 2.9.3. *Let $\mathcal{S}$ be any set of center-dangerous critical pairs whose signatures are identical. Then the set of critical pairs in $\mathcal{S}$ is reversible with one linear extension of $\mathbf{P}$.*

Proof. If $\pi^{\mathrm{in}}(a,b) = 1$ for all critical pairs in $\mathcal{S}$, then Lemma 2.9.2 implies that any alternating cycle in $\mathcal{S}$ must occur amongst critical pairs whose center-regions are identical, and then Theorem 2.7.4 implies that $\mathcal{S}$ is reversible. If instead $\pi^{\mathrm{in}}(a,b) = 0$ for all critical pairs in $\mathcal{S}$, then Proposition 2.7.5 implies that $\mathcal{S}$ is reversible. $\square$

The following corollary bounds the number of linear extensions of $\mathbf{P}$ that are needed to reverse all of center-dangerous critical pairs.

Corollary 2.9.4. *The center-dangerous critical pairs can be reversed with*

$$2h^3 R_2(3,h^2) R_3(h+1,4)^2$$

linear extensions of $\mathbf{P}$.

Proof. Theorem 2.9.3 implies that we only need to bound the number of signatures, each of which is composed of the same seven parameters. The parameters $|\,\mathrm{T}(\beta)|$,

$|P_\beta(a)|$, and $|P_\gamma(a)|$ can take on at most h distinct values since each represents the length of a directed path in $\mathbf{G}$. Theorem 2.6.11 implies that $\pi^{\mathrm{sp}}_{\beta,\gamma}(a) < R_2(3, h^2)$, Theorem 2.7.4 yields $\pi^{\mathcal{R}\mathrm{L}}(a, b)$ and $\pi^{\mathcal{R}\mathrm{R}}(a, b)$ are less than $R_3(h+1, 4)$, and of course $\pi^{\mathrm{in}}(a, b)$ takes on at most two distinct values. Combining these bounds gives the desired result. $\square$

2.10 The left and right regions

We start this section by considering the left-dangerous critical pairs. Recall from Section 2.8, that for each $a \in A$, we have identified a unimodal sequence that encodes the lengths of paths to certain special points. The special points that we will use to determine the regions for left-dangerous critical pairs are $\lambda_0(a)$, $\lambda_1(a)$, $\ldots$, $\Lambda(a)$. In particular, for each left-dangerous (a, b), there exists an index i such that $\lambda_{i+1}(a) <_T b <_T \lambda_i(a)$. For such a critical pair, we will define its *left-region* as $\mathfrak{R}_{\lambda_{i+1}(a),\lambda_i(a)}(a)$. As in Section 2.9, this notation is quite cumbersome. So we will instead refer to $\lambda_{i+1}(a)$ as $\alpha(a)$, or just α if the context is clear, and we will refer to $\lambda_i(a)$ as $\beta(a)$, or just β if the context is clear. Also, for the sake of notation in this section, we will write $\mathfrak{R}(a)$ instead of $\mathfrak{R}_{v,w}(a)$ and $m(a)$ instead of $m_{v,w}(a)$ when it is clear that we are referring to the left-region defined by the special points v and w.

While it is clear from the definition of a unimodal sequence, we feel that the following fact is significant enough to warrant emphasis.

Fact 2.10.1. Let (a, b) be a left-dangerous critical pair whose left-region is defined by special points α and β with $\alpha <_T \beta$. If $\gamma \in \mathrm{Spec}(a)$ such that $\alpha <_T \gamma <_T \beta$, then $|\,\mathrm{T}(\gamma)| \geq |\,\mathrm{T}(\alpha)|$. Furthermore, if $\delta \in \mathrm{Spec}(a)$ with $|\,\mathrm{T}(\delta)| < |\,\mathrm{T}(\alpha)|$, then $\delta \geq_T \beta$.

The next statement is intuitive, but we think it necessary to provide a few words of justification.

Proposition 2.10.2. *Let (a, b) be a left-dangerous critical pair whose left-region is* *defined by* $\alpha, \beta \in \mathrm{Spec}(a)$ *with* $\alpha <_T \beta$. *Let* $\gamma \in T$ *with* $|\,\mathrm{T}(\gamma)| \leq |\,\mathrm{T}(\alpha)|$ *and* $\gamma <_T \alpha$. *Then* γ *is embedded in* $\mathrm{Ex}(\mathfrak{R}(a))$.

Proof. Let w be minimal such that $w \in \mathrm{T}(\gamma) \cap \mathrm{T}(\alpha)$. Since $\gamma <_T \alpha$, the first edge on the path in T from w to γ is in $\mathrm{Ex}(\mathfrak{R}(a))$. If $\gamma \notin \mathrm{Ex}(\mathfrak{R}(a))$, then this path intersects $P_\alpha(a) \cup P_\beta(a)$. Thus $a < \gamma$ in $\mathbf{P}$, contrary to the choice of α. □

Recall that we use the notation $U(a)$ and $U[a]$ to denote the upset of a and the closed upset of a, respectively.

Lemma 2.10.3. *Let* (a, b) *and* (a', b') *be left-dangerous critical pairs. Let* α *and* β *be the special points that define* $\mathfrak{R}(a)$, *with* $\alpha <_T \beta$. *Let* $y \neq \alpha$ *be an element in* T *with* $y \leq_T \alpha$ *and* $|\,\mathrm{T}(y)| \leq |\,\mathrm{T}(\alpha)|$. *Let* $\mathfrak{W}$ *be an (unoriented) walk in* $\boldsymbol{G}$ *that satisfies the following:*

- *there is an* $x \in \partial(\mathfrak{R}(a)) \cap \mathrm{T'}(\beta)$ *such that* $\mathfrak{W} \cap \partial(\mathfrak{R}(a)) \subseteq \{x, y\}$,
- $\mathfrak{W} \cap \mathrm{T}(y) = \{y\}$ *and* $\mathfrak{W} \cap \mathrm{T}(x) = \{x\}$,
- a' *is comparable with every element in* $\mathfrak{W}$, *and*
- *every element in* $\mathfrak{W}$ *is less than* x *or* y *in* $\mathbf{P}$.

Then the unimodal sequence for a *is different than that for* a'.

Proof. By Proposition 2.10.2, $y \notin \mathrm{In}(\mathfrak{R}(a))$. Now consider the embedding of $\mathfrak{W}$. The first two items in the definition of $\mathfrak{W}$ and the fact that $y \notin \mathrm{In}(\mathfrak{R}(a))$ imply that $\mathfrak{W} \cap \mathrm{In}(\mathfrak{R}(a)) = \varnothing$. The union of $\mathfrak{W}$ and $\mathrm{T}(x, y)$ is a Jordan curve. Call this curve $\mathcal{C}$, and call the bounded and unbounded faces that $\mathcal{C}$ defines $\mathcal{F}$ and $\overline{\mathcal{F}}$, respectively. See Figure 2.17.

Observe that a is incomparable to all elements of $\mathbf{P}$ on $\mathcal{C}$; otherwise, we contradict the choice of α or the fact that $\beta \in \mathrm{Spec}(a)$. From the first item in the definition of $\mathfrak{W}$ and the fact that $\mathfrak{W} \cap \mathrm{In}(\mathfrak{R}(a)) = \varnothing$, it is clear that β is embedded in $\mathcal{F}$. Since

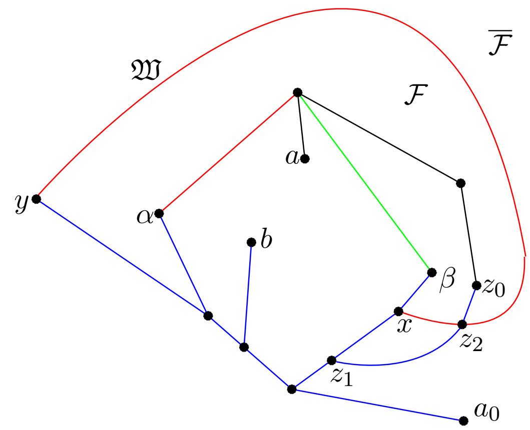


Figure 2.17: The curve $\mathcal{C}$ is composed of $\mathfrak{W}$ and $\mathrm{T}(x, y)$

$a < \beta$ in $\mathbf{P}$, we find, by the planarity of the embedding, that all elements in $U[a]$ are embedded in $\mathcal{F}$ as well.

Since $x \in U(a')$, we see that $\mathrm{Spec}(a')$ includes an element whose height in T is at most $|\,\mathrm{T}(x)|$. In particular, $\mathrm{Spec}(a')$ includes an element whose height in T is strictly less than $|\,\mathrm{T}(\beta)|$. If we assume that a and a' have the same unimodal sequences, then the element of $\mathrm{Spec}(a)$ with smallest height in T, say z_0, must have the same height as the element with smallest height in $\mathrm{Spec}(a')$. Notice that, by the definition of a unimodal sequence, $z_0 >_T \beta$. Let z_1 be the maximal element of $\mathrm{T}(z_0) \cap \mathrm{T}(\beta)$. Observe that $z_1 \in$ T'(x), since $|\,\mathrm{T}(z_0)| \leq |\,\mathrm{T}(x)|$. Therefore, the first edge on the directed path from z_1 to z_0 in T is embedded in $\overline{\mathcal{F}}$. However, $a < z_0$ in $\mathbf{P}$, and as such z_0 is embedded in $\mathcal{F}$. So, by the planarity of $\mathbf{G}$, there exists a point, say z_2, on $\mathcal{C}$ and on the path from z_1 to z_0 in T. Thus $z_2 < a'$ in $\mathbf{P}$, and the element of $\mathrm{Spec}(a')$ with smallest height in T has a height strictly less than $|\,\mathrm{T}(z_0)|$. Therefore a and a' have different unimodal sequences. $\square$

Proposition 2.10.4. *Let (a, b) and (a', b') be left-dangerous critical pairs whose left-regions are defined by the special points α, β and α', β', respectively, with $|\,\mathrm{T}(\alpha)| = |\,\mathrm{T}(\alpha')| > |\,\mathrm{T}(\beta)| = |\,\mathrm{T}(\beta')|$. Assume further that $\alpha <_T \alpha' <_T \beta$. Then either $\alpha' \in \mathrm{In}(\mathfrak{R}(a))$ or $\alpha' \in P_\beta(a)$ and $\beta' \leq_T \beta$.*

Proof. Suppose $\alpha' \notin \mathrm{In}(\mathfrak{R}(a))$. Then either $P_\beta(a)$ or $P_\alpha(a)$ intersects $\mathrm{T}(\alpha')$. Fact 2.10.1 and the fact that $\alpha, \beta \in \mathrm{Spec}(a)$ imply that $(P_\beta(a) \cup P_\alpha(a)) \cap \mathrm{T}(\alpha') \subseteq \{\alpha'\}$. So $\alpha' \in \partial(\mathfrak{R}(a))$. If $\alpha' \in P_\alpha(a)$, then $a' < \alpha$ in $\mathbf{P}$, contradicting the choice of α'. So $\alpha' \in P_\beta(a)$. Therefore $a' < \alpha' < \beta$ in $\mathbf{P}$, so the definition of a unimodal sequence for a' requires that $\beta' \leq_T \beta$, as desired. □

Lemma 2.10.5. *Let (a, b) and (a', b') be left-dangerous critical pairs. Let α, β and α', β' be the special points that define $\mathfrak{R}(a)$ and $\mathfrak{R}(a')$, respectively, and suppose that $|\,\mathrm{T}(\alpha)| = |\,\mathrm{T}(\alpha')| > |\,\mathrm{T}(\beta)| = |\,\mathrm{T}(\beta')|$. Further suppose that the unimodal sequence for a is the same as that for a'. Then it is not the case that $\alpha <_T \alpha' <_T \beta <_T \beta'$.*

Proof. Suppose not. Since $\beta <_T \beta'$, Proposition 2.10.4 implies $\alpha' \in \mathrm{In}(\mathfrak{R}(a))$. Now consider the embedding of $\mathfrak{R}(a)$. Since $\alpha <_T \alpha' <_T \beta <_T \beta'$ we find that $U[a'] \cap \partial(\mathfrak{R}(a)) = \varnothing$; otherwise, we contradict the choice of α' or β'. Since $\partial(\mathfrak{R}(a))$ is a Jordan curve we see that $U[a']$, and in particular a', $m(a')$, and β', are all embedded in $\mathrm{In}(\mathfrak{R}(a))$. From this we also see that $P_\beta(a)$ intersects $\mathrm{T}'(\beta')$. See Figure 2.18 for such a scenario.

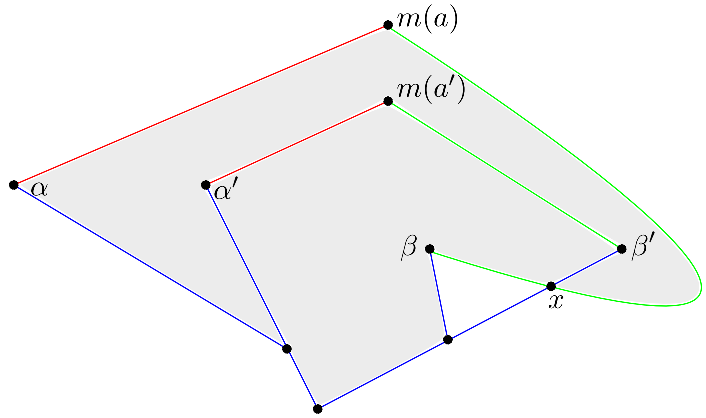


Figure 2.18: The shaded region is $\mathfrak{R}(a)$

Assume $P_\alpha(a) \cap \mathrm{T}(\beta') = \varnothing$. If we reverse the roles of a and a', set $y = \alpha$, and let $\mathfrak{W}$ be $P_\alpha(a)$ together with the subpath of $P_\beta(a)$ from $m(a)$ to the minimum point in $P_\beta(a) \cap \mathrm{T}(\beta')$, we get a contradiction from Lemma 2.10.3. So we may assume

$P_\alpha(a) \cap \mathrm{T}(\beta') \neq \varnothing$. In this case, let x be the maximum element of $\mathbf{P}$ in $P_\alpha(a) \cap \mathrm{T}(\beta')$, reverse the roles of a and a', set $y = \alpha$, and let $\mathfrak{W}$ be the subpath of $P_\alpha(a)$ from x to α. We have contradicted Lemma 2.10.3 again. $\square$

2.10.1 Fixing parameters

For the remainder of this section, our aim will be to disprove the existence of a strict alternating cycle (a_1, b_1), (a_2, b_2), $\ldots$, (a_k, b_k) with $a_{i-1} < b_i$ cyclically, such that each pair in the cycle is identical with respect to a set of parameters. The first three parameters are:

(1) (a_i, b_i) is left-dangerous for all $i \in [k]$,

(2) the unimodal sequence for a_i is identical to that for a_j for all $i, j \in [k]$, and

(3) there are integers k_1 and k_2 such that $k_1 = |\,\mathrm{T}(\alpha_i)| > |\,\mathrm{T}(\beta_i)| = k_2$ for all $i \in [k]$, where α_i and β_i are the special points that define the left-region $\mathfrak{R}(a_i)$.

Later in this section we will add the remaining parameters.

For the sake of notation, let $\mathcal{P}(a_{j-1}, b_j)$ denote any fixed oriented path in $\mathbf{G}$ from a_{j-1} to b_j, let $\mathcal{P}(a_j, \alpha_j)$ denote any fixed oriented path in $\mathbf{G}$ from a_j to α_j, and let $\mathcal{P}(a_j, \beta_j)$ denote any fixed oriented path in $\mathbf{G}$ from a_j to β_j. We will write m_j instead of $m(a_j)$. Also, unless stated otherwise, we will set $a = a_i$ and $a' = a_{i-1}$ in all future applications of Lemma 2.10.3, and thus not state that we are doing so in the proofs.

2.10.2 Going left

The purpose of the next few lemmas is to examine the case in which there exists an index i such that $\alpha_{i-1} <_T \alpha_i$.

Lemma 2.10.6. *Let $\alpha_{i-1} <_T \alpha_i$. Suppose $\mathcal{P}(a_{i-1}, \alpha_{i-1}) \cap \mathrm{T}'(\alpha_i) = \varnothing$. Then $a_{i-1} \in \mathrm{Ex}(\mathfrak{R}(a_i))$, and the minimum element in $\mathcal{P}(a_{i-1}, b_i) \cap \partial(\mathfrak{R}(a_i))$ is in $\mathrm{T}'(\alpha_i)$.*

Proof. Assume $a_{i-1} \notin \mathrm{Ex}(\mathfrak{R}(a_i))$. Then $a_{i-1} \in \mathrm{In}(\mathfrak{R}(a_i))$, since clearly a_{i-1} is not on the boundary of $\mathfrak{R}(a_i)$. Proposition 2.10.2 says $\alpha_{i-1} \in \mathrm{Ex}(\mathfrak{R}(a_i))$, so $\mathcal{P}(a_{i-1}, \alpha_{i-1}) \cap \partial(\mathfrak{R}(a_i)) \subseteq \mathrm{T'}(\beta_i)$, as else we contradict the choice of α_i. Let x be maximal in $\mathbf{P}$ such that $x \in \mathcal{P}(a_{i-1}, \alpha_{i-1}) \cap \mathrm{T}(\beta_i)$ and define $\mathcal{P}(x, \alpha_{i-1})$ to be the subpath of $\mathcal{P}(a_{i-1}, \alpha_{i-1})$ from x to α_{i-1}. See Figure 2.19a. Setting $\mathfrak{W} = \mathcal{P}(x, \alpha_{i-1})$ and $y = \alpha_{i-1}$ in order to apply Lemma 2.10.3, we find that the unimodal sequences for a_i and a_{i-1} differ, a contradiction.

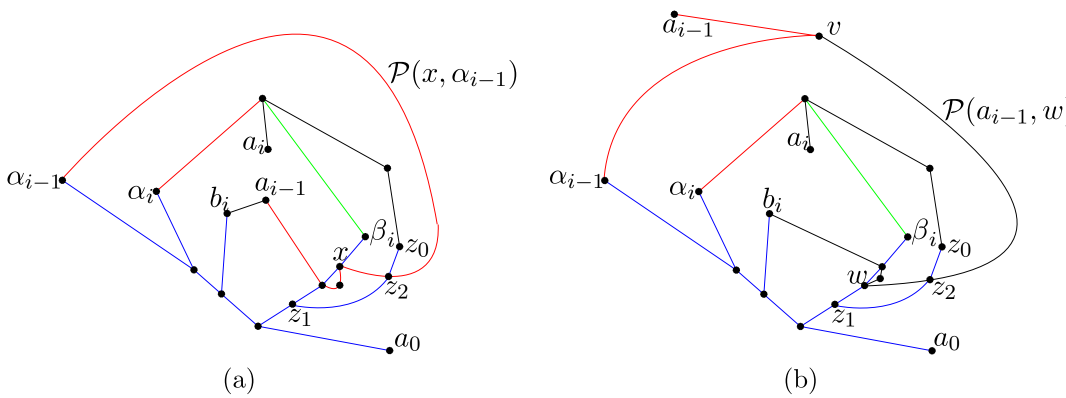


Figure 2.19

So $a_{i-1} \in \mathrm{Ex}(\mathfrak{R}(a_i))$. Since $a_{i-1} < b_i$ in $\mathbf{P}$, it is clear that the minimum element in $\mathcal{P}(a_{i-1}, b_i) \cap \partial(\mathfrak{R}(a_i))$, say w, exists. Note that w cannot be greater than a_i in $\mathbf{P}$, since $w < b_i$, which implies $w \in T$. By way of contradiction, assume that $w \in \mathrm{T'}(\beta_i)$. Define $\mathcal{P}(a_{i-1}, w)$ to be the subpath of $\mathcal{P}(a_{i-1}, b_i)$ from a_{i-1} to w. See Figure 2.19b.

If $\mathcal{P}(a_{i-1}, \alpha_{i-1}) \cap \mathrm{T'}(\beta_i) \neq \varnothing$, then set x as the maximum element in this intersection, $\mathfrak{W}$ as the subpath of $\mathcal{P}(a_{i-1}, \alpha_{i-1})$ from x to α_{i-1}, and $y = \alpha_{i-1}$ in order to apply Lemma 2.10.3 and reach a contradiction. Otherwise, let v be the maximal element in $\mathcal{P}(a_{i-1}, \alpha_{i-1}) \cap \mathcal{P}(a_{i-1}, w)$ and set $\mathfrak{W}$ as the subpath of $\mathcal{P}(a_{i-1}, w)$ from v to w together with the subpath of $\mathcal{P}(a_{i-1}, \alpha_{i-1})$ from v to α_{i-1}, $x = w$, and $y = \alpha_{i-1}$ in order to apply Lemma 2.10.3 and reach the same contradiction. □

Lemma 2.10.7. *If $\alpha_{i-1} <_T \alpha_i$ then $\beta_{i-1} \leq_T \alpha_i$.*

Proof. Assume first that a_{i-1} is not embedded in $\mathrm{Ex}(\mathfrak{R}(a_i))$. By Lemma 2.10.6, we know that $\mathcal{P}(a_{i-1}, \alpha_{i-1}) \cap \mathrm{T'}(\alpha_i) \neq \varnothing$. Since α_i cannot be in $\mathcal{P}(a_{i-1}, \alpha_{i-1})$ without contradicting the choice of α_i, there must exist a $\gamma \in \mathrm{Spec}(a_{i-1})$ with $|\,\mathrm{T}(\gamma)| < |\,\mathrm{T}(\alpha)|$ and $\gamma \in \mathrm{T'}(\alpha_i)$. By Fact 2.10.1, we find that $\beta_{i-1} \leq_T \gamma$, and so $\beta_{i-1} \leq_T \alpha_i$.

So we may assume that a_{i-1} is embedded in $\mathrm{Ex}(\mathfrak{R}(a_i))$. By Fact 2.10.1, we are done if $\mathcal{P}(a_{i-1}, \alpha_{i-1}) \cap \mathrm{T'}(\alpha_i) \neq \varnothing$, so we may assume otherwise. By Lemma 2.10.6, $\mathcal{P}(a_{i-1}, b_i) \cap \mathrm{T'}(\alpha_i) \neq \varnothing$. Then using the same argument as in the previous case, we arrive at our desired result. □

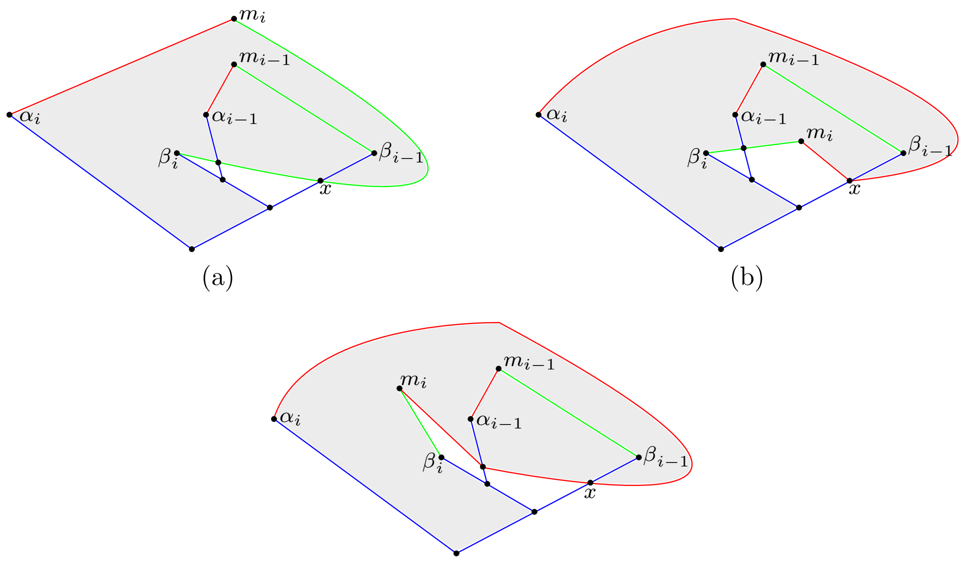
2.10.3 Going right

The purpose of the next lemma is to examine the case in which there exists an index i such that $\alpha_{i-1} >_T \alpha_i$.

Lemma 2.10.8. *Let $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ with $a_{i-1} < b_i$, cyclically, be an alternating cycle such that each critical pair in the cycle satisfies conditions (1)–(3) above. If $\alpha_{i-1} >_T \alpha_i$, then $\beta_{i-1} \leq_T \beta_i$.*

Proof. Consider the embedding of $\mathfrak{R}(a_i)$. Assume for the moment that a_{i-1} is embedded in $\mathrm{Ex}(\mathfrak{R}(a_i))$. Since $a_{i-1} < b_i$, we must have $\mathcal{P}(a_{i-1}, b_i) \cap \partial(\mathfrak{R}(a_i)) \neq \varnothing$. If these paths intersect in $P_{\alpha_i}(a_i) \cup P_{\beta_i}(a_i)$, then we contradict the fact that $a_i \| b_i$ in $\mathbf{P}$. If the paths intersect in $\mathrm{T}(\alpha_i)$, then α_{i-1} cannot define the left-region for a_{i-1}. Thus, the intersection must occur in $\mathrm{T'}(\beta_i)$. Since the unimodal sequences for a_i and a_{i-1} are identical, we find that $\alpha_{i-1} <_T \beta_{i-1} <_T \beta_i$, as desired. So we may assume that a_{i-1} is embedded in $\mathrm{In}(\mathfrak{R}(a_i))$.

Let $x_1 \in U(a_{i-1})$. If x_1 is not embedded in $\mathrm{In}(\mathfrak{R}(a_i))$, then there must be some point, say x_2, in $U(a_{i-1}) \cap \partial(\mathfrak{R}(a_i))$. We may assume $x_2 \not< \beta_i$ in $\mathbf{P}$; otherwise $\beta_{i-1} \leq_T \beta_i$. But then $x_2 < \alpha_i$ in $\mathbf{P}$, contrary to the assumption that $\alpha_{i-1} >_T \alpha_i$. Therefore, we may assume that all points in $U[a_{i-1}]$ are embedded in $\mathrm{In}(\mathfrak{R}(a_i))$. In particular α_{i-1}, β_{i-1}, and m_{i-1} are embedded in $\mathrm{In}(\mathfrak{R}(a_i))$.



(a) (b) (c)

Figure 2.20: The shaded areas are $\mathfrak{R}(a_i)$

By Lemma 2.10.5, we are done if we can show that $\alpha_{i-1} <_T \beta_i$. So assume not. Thus $\beta_{i-1} >_T \beta_i$, and as such the edge closest to a_0 in $\mathrm{T}(\alpha_{i-1}) \setminus \mathrm{T}(\beta_i)$ is embedded in the unbounded face of the plane defined by $\partial(\mathfrak{R}(a_i))$. Since $\beta_{i-1} \in \mathrm{In}(\mathfrak{R}(a_i))$, there must be a nonempty intersection between T'(β_{i-1}) and $P_{\alpha_i}(a_i) \cup P_{\beta_i}(a_i)$. Let x be the maximum element of $\mathbf{P}$ in this intersection. Let y be maximal with respect to the linear order on T such that $y \leq_T \alpha_{i-1}$, $|\,\mathrm{T}(y)| \leq |\,\mathrm{T}(\alpha_{i-1})|$, and y is on the subwalk of $P_{\alpha_i}(a_i) \cup P_{\beta_i}(a_i)$ from x to α_i. Notice that y exists, since α_i satisfies the relevant criteria, and $y \neq \alpha_{i-1}$; otherwise a_{i-1} is less than α_i or β_i, contrary to the choices of α_{i-1} and β_{i-1}, respectively. See Figure 2.20 for examples of such a scenario. Then, setting $a = a_{i-1}$, $a' = a_i$, and $\mathfrak{W}$ to be the subwalk of $P_{\alpha_i}(a_i) \cup P_{\beta_i}(a_i)$ from x to y, we contradict Lemma 2.10.3. $\square$

2.10.4 A new parameter

For each left-dangerous critical pair (a, b), define a q-*sequence starting at* (a, b) as a list of left-dangerous critical pairs $(a, b) = (a_1, b_1), (a_2, b_2), \ldots, (a_t, b_t)$, with special points $\alpha_j <_T \beta_j$ defining the left-region $\mathfrak{R}(a_j)$, such that for all $j \in [t-1]$:

- $\alpha_{j+1} = \alpha_j$,
- $\beta_{j+1} >_T \beta_j$, and
- m_{j+1} is embedded in $\mathrm{In}(\mathfrak{R}(a_j))$,

and such that

- there is an integer k_2 such that $|\,\mathrm{T}(\alpha_j)| > |\,\mathrm{T}(\beta_j)| = k_2$ for all $j \in [t]$.

For each left-dangerous (a, b) define the parameter $\pi^{\mathrm{q}}(a, b)$ to be the longest q-sequence starting at (a, b). We wish to upper bound this parameter with some function of h. To this end, let $t \in \mathbb{N}$ be sufficiently large such that any partition of the two-element subsets of $[t]$ into $1 + 2h^2$ classes results in a subset of size $h+1$ such that all two-element subsets of this $(h+1)$-set are in the first class, or results in a subset of size 3 such that all two-element subsets of this 3-set are in some class other than the first. Denote the minimum such t by $R_2(h + 1, 1; 3, 2h^2)$, which exists by Ramsey's theorem.

Lemma 2.10.9. *Let (a, b) be a left-dangerous critical pair whose left-region is defined by special points α and β with $\alpha <_T \beta$. Then $\pi^{\mathrm{q}}(a, b) < R_2(h + 1, 1; 3, 2h^2)$.*

Proof. Let $(a, b) = (a_1, b_1)$, (a_2, b_2), $\ldots$, (a_t, b_t) be the longest q-sequence starting at (a, b). Notice that the requirements of a q-sequence force $\alpha_i = \alpha_j$ and $\beta_j >_T \beta_i$ for all $1 \le i < j \le t$. We will refer to the common special point as α.

We have the following claim: $m_j \in \mathrm{In}(\mathfrak{R}(a_i))$ for all $1 \le i < j \le t$. To show this, we fix any index i and use induction on j. For $j = i + 1$, the claim is true by the definition of q-sequence. So assume the claim has been verified for some $j \ge i + 1$. Notice that $m_j \| x$ for all $x \in \mathrm{T'}(\alpha, \beta_i) \cup P_{\beta_i}(a_i)$; otherwise, we contradict the fact that $\alpha \in \mathrm{Spec}(a_j)$ or the choice of β_j in defining $\mathfrak{R}(a_j)$. Therefore, by Corollary 2.6.6, we know $P_\alpha(a_j) \cap \mathrm{Ex}(\mathfrak{R}(a_i)) = \varnothing$. Now consider the location of m_{j+1} in the embedding. By assumption, $m_{j+1} \in \mathrm{In}(\mathfrak{R}(a_j))$, so the claim fails only if

$m_{j+1} \in \text{In}(\mathfrak{R}(a_j)) \setminus \text{In}(\mathfrak{R}(a_i))$, and we'll assume this is the case. As we will let the reader verify, this only happens if m_{j+1} is in the bounded face defined by a Jordan curve $\mathcal{C}$ that satisfies the following: (1) all elements of $\mathbf{P}$ on $\mathcal{C}$ are less than either β_i or β_j in $\mathbf{P}$, and (2) α is not in the bounded face defined by $\mathcal{C}$. Therefore, any directed path from m_{j+1} to α must hit an element of $\mathbf{P}$ on $\mathcal{C}$, contrary to the choice of β_{j+1}, and concluding the proof of the claim. See Figure 2.21 for an example with $j+1 = i+2 = 3$.

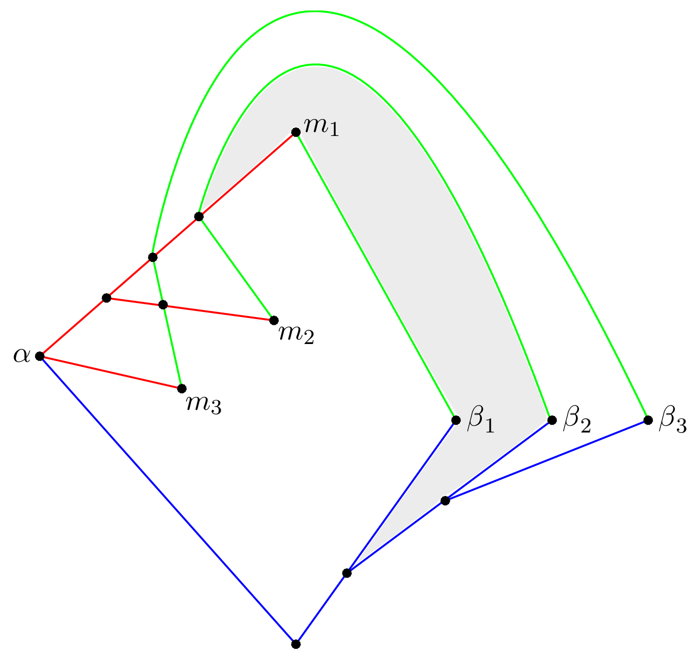


Figure 2.21: A q-sequence starting at (a_1, b_1) of length three. Each ordered pair here would receive color 1.

We continue the proof by coloring the ordered pairs (i, j) for $1 \leq i < j \leq t$. Give (i, j) color 1 if $\beta_j \notin \text{In}(\mathfrak{R}(a_i))$. If (i, j) does not receive color 1 we know $\beta_j \in \text{In}(\mathfrak{R}(a_i))$. In this case, $\text{T'}(\beta_j) \cap \partial(\mathfrak{R}(a_i))$ is nonempty. Let $\text{z}_{i,j}$ be the maximum element of $\mathbf{P}$ in this intersection. Give (i, j) color $2_{k,l}$ if $\text{z}_{i,j} \in P_\alpha(a_i)$, $|\,\text{T}(\text{z}_{i,j})| = k$, and the subpath of $P_\alpha(a_i)$ from m_i to $\text{z}_{i,j}$ has length l. Similarly, give (i, j) color $3_{k,l}$ if $\text{z}_{i,j} \in P_{\beta_i}(a_i)$, $|\,\text{T}(\text{z}_{i,j})| = k$, and the subpath of $P_{\beta_i}(a_i)$ from m_i to $\text{z}_{i,j}$ has length l. Notice that $1 \leq k, l \leq h$, and as such we have used a total of $1 + 2h^2$ colors.

Assume $t \geq R_2(h+1, 1; 3, 2h^2)$, and let color 1 be the distinguished color class in the definition of this Ramsey number. Suppose first that there is a set A of size at least $h+1$ whose two-element subsets have color 1. Keep the first $h+1$ of these pairs and relabel them (a_1, b_1), (a_2, b_2), $\ldots$, (a_{h+1}, b_{h+1}). Consider the ordered pairs $(1, j)$ for $2 \le j \le h+1$. Recall that $m_j \| x$ in $\mathbf{P}$ for all $x \in \mathrm{T'}(\alpha, \beta_1) \cup P_{\beta_1}(a_1)$. Thus, since $\beta_j \notin \mathrm{In}(\mathfrak{R}(a_1))$, the intersection between $P_\alpha(a_1)$ and $P_{\beta_j}(a_j)$ is nonempty. Let the maximal element of $\mathbf{P}$ in this intersection be denoted y_j and set $y_1 = m_1$. Since the y_j are all on $P_\alpha(a_1)$, and since $\beta_j >_T \beta_i$, we find that $y_j > y_i$ in $\mathbf{P}$ for all $1 \le i < j \le h+1$. Therefore $P_\alpha(a_1)$ has length at least $h+1$, a contradiction.

Next suppose that the ordered pairs in A each have color $2_{k,l}$, for some $1 \le k, l \le h$ and that $|A| \ge 3$. Keep the first 3 of these pairs and relabel them (a_1, b_1), (a_2, b_2), (a_3, b_3). Since $|\,\mathrm{T}(\mathrm{z}_{1,3})| = k = |\,\mathrm{T}(\mathrm{z}_{2,3})|$ we find that $\mathrm{z}_{1,3} = \mathrm{z}_{2,3}$. Further, since the subpaths of $P_\alpha(a_i)$ from m_1 to $\mathrm{z}_{1,2}$ and from m_1 to $\mathrm{z}_{1,3}$ each have length l, we find that $\mathrm{z}_{1,2} = \mathrm{z}_{1,3}$. Therefore, $\mathrm{z}_{1,2} = \mathrm{z}_{1,3} = \mathrm{z}_{2,3}$, which implies $\mathrm{z}_{2,3} \in \mathrm{T'}(\beta_2)$. But $m_2 < \mathrm{z}_{2,3}$ in $\mathbf{P}$, contradicting the fact that $\beta_2 \in \mathrm{Spec}(a_2)$. See Figure 2.22.

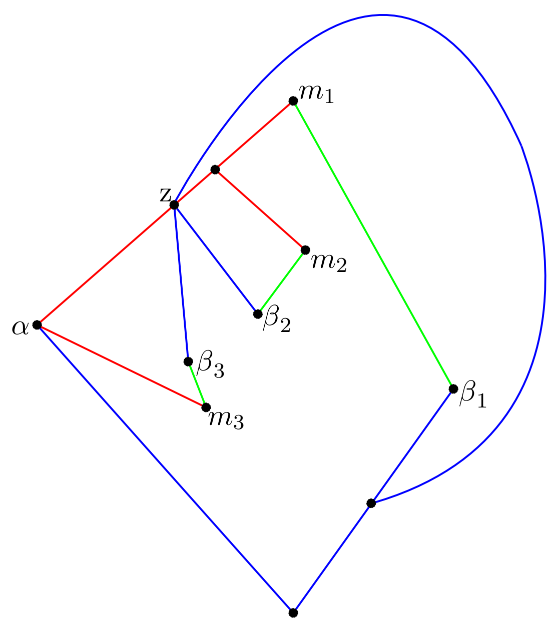


Figure 2.22: Here $\mathrm{z} = \mathrm{z}_{1,2} = \mathrm{z}_{1,3} = \mathrm{z}_{2,3}$. We see $z \in \mathrm{T'}(\beta_2) \cap P_\alpha(a_2)$, a contradiction.

Finally, suppose that the ordered pairs in A each have color $3_{k,l}$ for some $1 \le k, l \le h$ and that $|A| \ge 3$. Keep the first 3 of these pairs and relabel them (a_1, b_1),

(a_2, b_2), (a_3, b_3). An analogous argument to the previous case yields the analogous contradiction. $\square$

2.10.5 Bounding the number of signatures for left-dangerous pairs

To conditions (1)–(3) on our alternating cycle, add the following conditions:

(4) $|P_{\alpha_i}(a_i)| = |P_{\alpha_j}(a_j)|$ for all $i, j \in [k]$, and

(5) $\pi^{\mathrm{q}}(a_i, b_i) = \pi^{\mathrm{q}}(a_j, b_j)$ for all $i, j \in [k]$.

The tools that we have developed thus far allow us to prove the following lemma, which states that the pairs of special points that define the regions for the critical pairs in the alternating cycle are all the same.

Lemma 2.10.10. *Let $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ with $a_{i-1} < b_i$, cyclically, be a strict alternating cycle such that each critical pair in the cycle satisfies conditions (1)–(5) above. Then each left-region is determined by the same two special points; that is, $\alpha_i = \alpha_j$ and $\beta_i = \beta_j$ for all $i, j \in [k]$.*

Proof. Set β to be the element of $\{\beta_j\}_{j=1}^{k}$ that is least in the linear order on T. Amongst those critical pairs in the alternating cycle that use β to define its left-region, select one whose other special point is greatest in the linear order on T. Call this point α, and let (a_i, b_i) be this chosen critical pair; that is, $\alpha_i = \alpha$ and $\beta_i = \beta$.

Consider α_{i-1}. Since the pairs in the alternating cycle satisfy conditions (1), (2), and (2), Lemma 2.10.7 tells us that if $\alpha_{i-1} <_T \alpha_i$, then $\beta_{i-1} <_T \alpha_i$ as well. So the choice of β_i implies that $\alpha_{i-1} \not<_T \alpha_i$. Lemma 2.10.8 tells us that if $\alpha_{i-1} >_T \alpha_i$, then $\beta_{i-1} \leq_T \beta_i$. Therefore, the choice of both β and α implies that $\alpha_{i-1} \not>_T \alpha_i$. So $\alpha_{i-1} = \alpha$.

Now turn to β_{i-1}. The choice of β yields $\beta_{i-1} \geq_T \beta_i$. We would like $\beta_{i-1} = \beta_i$, so suppose otherwise, that $\beta_{i-1} >_T \beta_i$. Further suppose that $a_{i-1} \in \mathrm{Ex}(\mathfrak{R}(a_i))$ and consider the embedding of $\mathcal{P}(a_{i-1}, b_i)$. As $b_i \in \mathrm{In}(\mathfrak{R}(a_i))$, we see that the intersection of $\mathcal{P}(a_{i-1}, b_i)$ and $\partial(\mathfrak{R}(a_i))$ is nonempty. Let x be in this intersection. Clearly x is not in $P_\alpha(a_i)$ or $P_\beta(b_i)$; otherwise $a_i < b_i$ in $\mathbf{P}$. Furthermore, $x \notin$ T'(β) by the choice that β_{i-1} defines $\mathfrak{R}(a_{i-1})$, and $x \notin$ T'(α), since $\alpha \in \mathrm{Spec}(a_i)$. Thus x does not exist, and we conclude that a_{i-1} is embedded in $\mathrm{In}(\mathfrak{R}(a_i))$.

Suppose m_{i-1} is embedded in $\mathrm{Ex}(\mathfrak{R}(a_i))$. Both $U[a_{i-1}]$ and $U[m_{i-1}]$ cannot intersect T'(α, β) nor $P_\beta(a_i)$; otherwise, we contradict the fact that $\alpha \in \mathrm{Spec}(a_{i-1})$ or the choice of β_{i-1} in defining $\mathfrak{R}(a_{i-1})$. Thus, since $a_{i-1} \in \mathrm{In}(\mathfrak{R}(a_i))$, we find that $\mathcal{P}(a_{i-1}, m_{i-1}) \cap \partial(\mathfrak{R}(a_i))$ is a nonempty subset of $P_\alpha(a_i)$. Let y be the maximum element of $\mathbf{P}$ in this intersection. Also, we find that $P_\alpha(a_{i-1}) \cap \partial(\mathfrak{R}(a_i))$ is a nonempty subset of $P_\alpha(a_i)$, and by Corollary 2.6.6, $P_\alpha(a_{i-1}) \cap \mathrm{In}(\mathfrak{R}(a_i)) = \varnothing$. Let z be the minimum element of $\mathbf{P}$ in $P_\alpha(a_{i-1}) \cap \partial(\mathfrak{R}(a_i))$. If the subpath of $\mathcal{P}(a_{i-1}, m_{i-1})$ from y to m_{i-1} intersects $P_\alpha(a_{i-1})$ or $P_{\beta_{i-1}}(a_{i-1})$, then $\mathbf{G}$ has a directed cycle. So we may assume that $\mathcal{P}(a_{i-1}, m_{i-1}) - m_{i-1}$ is in the interior of $\mathfrak{R}(a_{i-1})$, and, in particular, that $y < z$ in $\mathbf{P}$. But then, regardless of the location of β_{i-1}, we can find a region that contradicts the minimality of $\mathfrak{R}(a_{i-1})$ (using y as the new m_{i-1}). See Figure 2.23.

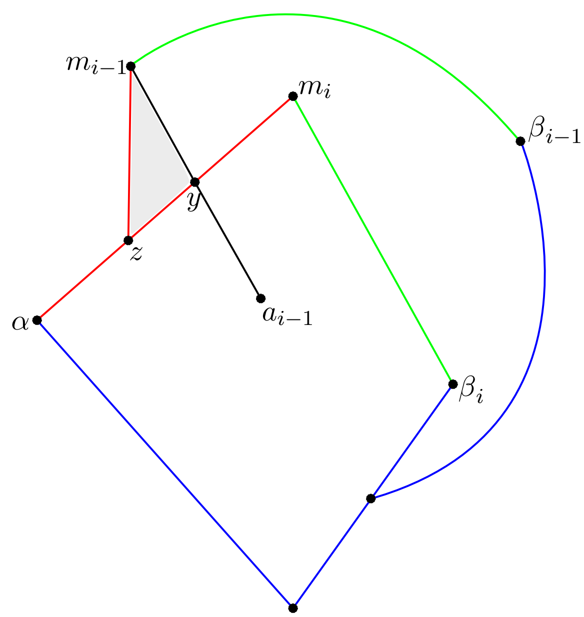


Figure 2.23: After deleing the shaded area we are left with a region that contradicts the minimality of $\mathfrak{R}(a_{i-1})$

Next suppose that m_{i-1} is embedded in $\partial(\mathfrak{R}(a_i))$. By the logic above, m_{i-1} must be in $P_\alpha(a_i) - m_i$. However, this contradicts condition (4) of the alternating cycle, since it implies that $|P_\alpha(a_i)| > |P_\alpha(a_{i-1})|$. So we may assume that m_{i-1} is embedded in $\mathrm{In}(\mathfrak{R}(a_i))$. But this contradicts condition (5) on the alternating cycle, since it implies that $\pi^{\mathrm{q}}(a_i, b_i) \geq \pi^{\mathrm{q}}(a_{i-1}, b_{i-1}) + 1$.

Therefore $\beta_{i-1} = \beta_i$. Applying the above arguments to α_{i-2} and β_{i-2}, then α_{i-3} and β_{i-3}, and so on, we eventually arrive at the desired result. □

To conditions (1)–(5), we add the following conditions on our alternating cycle:

(6) $|P_{\beta_i}(a_i)| = |P_{\beta_j}(a_j)|$ for all $i, j \in [k]$, and

(7) $\pi^{\mathrm{sp}}_{\alpha,\beta}(a_i, b_i) = \pi^{\mathrm{sp}}_{\alpha,\beta}(a_j, b_j)$ for all $i, j \in [k]$.

The next theorem allows us to assume that all critical pairs in our alternating cycle have identical left-regions.

Lemma 2.10.11. *Let $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ with $a_{i-1} < b_i$, cyclically, be a strict alternating cycle such that each critical pair in the cycle satisfies conditions (1)–(7) above. Then $\mathfrak{R}(a_i)$ and $\mathfrak{R}(a_j)$ are identical for all $i, j \in [k]$.*

Proof. Lemma 2.10.10 allows us to assume that the left-region for each critical pair in the alternating cycle is defined by α and β, with $|\mathrm{T}(\alpha)| > |\mathrm{T}(\beta)|$. Condition (7) and Theorem 2.6.11 then imply that all of the left-regions are inclusion-wise comparable. Let $\mathfrak{R}(a_i)$ be a minimal element in this inclusion-wise order. We wish to show that $\mathfrak{R}(a_{i-1})$ is identical to $\mathfrak{R}(a_i)$. If we can do so, we can then apply the same analysis to $\mathfrak{R}(a_{i-2})$, $\mathfrak{R}(a_{i-3})$, and so on, we arrive at the desired result.

Suppose a_{i-1} is embedded in $\mathrm{Ex}(\mathfrak{R}(a_i))$. Since $b_i \in \mathrm{In}(\mathfrak{R}(a_i))$, we find that $\mathcal{P}(a_{i-1}, b_i)$ intersects $\partial(\mathfrak{R}(a_i))$. However, any such intersection contradicts the fact that $\alpha, \beta \in \mathrm{Spec}(a_{i-1})$ or that $a_i \| b_i$ in $\mathbf{P}$. So $a_{i-1} \in \mathrm{In}(\mathfrak{R}(a_i))$.

Next suppose that m_{i-1} is embedded in $\mathrm{Ex}(\mathfrak{R}(a_i))$. Since $a_{i-1} \in \mathrm{In}(\mathfrak{R}(a_i))$, we see that $\mathcal{P}(a_{i-1}, m_{i-1})$ intersects $\partial(\mathfrak{R}(a_i))$. Let y be the greatest element of $\mathbf{P}$ in this intersection. Since all regions are inclusion-wise comparable, we see that neither $P_\alpha(a_{i-1})$ nor $P_\beta(a_{i-1})$ intersect $\mathrm{In}(\mathfrak{R}(a_i))$. Let z_α be the least element of $\mathbf{P}$ in $P_\alpha(a_{i-1}) \cap P_\alpha(a_i)$, and let z_β be the least element of $\mathbf{P}$ in $P_\beta(a_{i-1}) \cap P_\beta(a_i)$. If the subpath of $\mathcal{P}(a_{i-1}, m_{i-1})$ from y to m_{i-1} intersects $P_\alpha(a_{i-1})$ or $P_\beta(a_{i-1})$, then $\mathbf{G}$ has a directed cycle. So we may assume that $\mathcal{P}(a_{i-1}, m_{i-1}) - m_{i-1}$ is in the interior of $\mathfrak{R}(a_{i-1})$. As such, we can find a region that contradicts the minimality of $\mathfrak{R}(a_{i-1})$ (using y as the new m_{i-1}).

Now suppose m_{i-1} is embedded in $\mathrm{In}(\mathfrak{R}(a_i))$. Then, since all regions are ordered by inclusion, $\mathfrak{R}(a_{i-1})$ is strictly contained $\mathfrak{R}(a_i)$, contrary to the assumption that $\mathfrak{R}(a_i)$ is minimal in the inclusion-wise order on the regions. So we may assume that m_{i-1} is on $\partial(\mathfrak{R}(a_i))$. If $m_{i-1} \neq m_i$, then we contradict condition (4) or (6) of the alternating cycle. Therefore $m_{i-1} = m_i$, which, by Corollary 2.6.6, implies that $\mathfrak{R}(a_{i-1})$ and $\mathfrak{R}(a_i)$ are identical. $\square$

For left-dangerous critical pairs, the parameter $\pi^{\mathrm{in}}(a, b)$ is defined to be 1 if a is embedded in $\mathrm{In}(\mathfrak{R}_{\alpha,\beta}(a))$ and to be 0 otherwise. For every left-dangerous critical pair (a, b), define $\Sigma(a, b)$ as the vector with the following coordinates:

- the unimodal sequence for a,
- $|\,\mathrm{T}(\alpha)|$,
- $|\,\mathrm{T}(\beta)|$,
- $|P_\alpha(a)|$,
- $|P_\beta(a)|$,
- $\pi^{\mathrm{q}}(a, b)$,

- $\pi^{\mathrm{sp}}_{\alpha,\beta}(a)$,
- $\pi^{\mathcal{R}\mathrm{L}}(a,b)$,
- $\pi^{\mathcal{R}\mathrm{R}}(a,b)$, and
- $\pi^{\mathrm{in}}(a,b)$,

where we have substituted $\mathcal{R}$ for $\mathfrak{R}_{\alpha,\beta}(a)$ to simplify notation. We can now prove the main result of this section.

Theorem 2.10.12. *Let $\mathcal{S}$ be any set of left-dangerous critical pairs whose signatures are identical. Then the set of critical pairs in $\mathcal{S}$ is reversible with one linear extension of $\mathbf{P}$.*

Proof. If $\pi^{\mathrm{in}}(a,b) = 1$ for all critical pairs in $\mathcal{S}$, then Lemma 2.10.11 implies that any alternating cycle in $\mathcal{S}$ must occur amongst critical pairs whose left-regions are identical, and then Theorem 2.7.4 implies that $\mathcal{S}$ is reversible. If instead $\pi^{\mathrm{in}}(a,b) = 0$ for all critical pairs in $\mathcal{S}$, then Proposition 2.7.5 implies that $\mathcal{S}$ is reversible. $\square$

The following corollary bounds the number of linear extensions of $\mathbf{P}$ that are needed to reverse all of left-dangerous critical pairs.

Corollary 2.10.13. *The left-dangerous critical pairs can be reversed with*

$$2^{2h+1}h^4R_2(h+1,1;3,2h^2)R_2(3,h^2)R_3(h+1,4)^2$$

linear extensions of $\mathbf{P}$.

Proof. Theorem 2.10.12 implies that we only need to bound the number of signatures, each of which is composed of the same ten parameters. The parameters $|\,\mathrm{T}(\alpha)|$, $|\,\mathrm{T}(\beta)|$, $|P_\beta(a)|$, and $|P_\gamma(a)|$ can take on at most h distinct values, since each represents a length of a directed path in $\mathbf{G}$. (We can save a factor of 2 by using the fact that $|\,\mathrm{T}(\alpha)| > |\,\mathrm{T}(\beta)|$, but this makes no difference given the enormity of the other parameters, so for simplicity we have ignored it.) Proposition 2.8.2 says that the number of distinct unimodal sequences is 2^{2h}, Theorem 2.10.9 gives $\pi^{\mathrm{q}}(a,b) < R_2(h+1,1;3,2h^2)$, Theorem 2.6.11 implies that $\pi^{\mathrm{sp}}_{\alpha,\beta}(a) < R_2(3,h^2)$, Theorem 2.7.4 yields $\pi^{\mathcal{R}\mathrm{L}}(a,b)$ and $\pi^{\mathcal{R}\mathrm{R}}(a,b)$ are less than $R_3(h+1,4)$, and of course $\pi^{\mathrm{in}}(a,b)$ takes on at most two distinct values. Combining these bounds gives the desired result. $\square$

2.10.6 Right-dangerous critical pairs

The special points that we will use to determine the regions for right-dangerous critical pairs are $\omega_0(a)$, $\omega_1(a)$, $\ldots$, $\Omega(a)$. In particular, for each right-dangerous (a,b) there exists a $0 \le i \le h-1$ such that $\omega_i(a) <_T b <_T \omega_{i+1}(a)$. For such a critical pair we will define its *right-region* as $\mathfrak{R}_{\omega_i(a),\omega_{i+1}(a)}(a)$.

It should be clear that all of the work we did to bound the number of signatures of left-dangerous critical pairs applies analogously to the right-dangerous critical pairs. In particular, for any right-dangerous critical pair (a,b) we can define $\Sigma(a,b)$ by substituting $\omega_i(a)$ for β and $\omega_{i+1}(a)$ for α in the definition of the signature of left-dangerous critical pairs, and obtain the following theorem and corollary.

Theorem 2.10.14. *Let $\mathcal{S}$ be any set of right-dangerous critical pairs whose signatures are identical. Then the set of critical pairs in $\mathcal{S}$ is reversible with one linear extension of* $\mathbf{P}$.

Corollary 2.10.15. *The right-dangerous critical pairs can be reversed with*

$$2^{2h+1}h^4 R_2(h+1,1;3,2h^2)R_2(3,h^2)R_3(h+1,4)^2$$

linear extensions of $\mathbf{P}$.

2.11 A bound in the special case

From Section 2.5 until now we have been working within the following *special case*: there is an element $a_0 \in \min(\mathbf{P})$ such that $a_0 < b$ in $\mathbf{P}$, for every element $b \in \max(\mathbf{P})$. We are now prepared to state the analog of Theorem 2.5.1 in this special case.

Theorem 2.11.1. *For every $h \geq 1$, there exists a least positive integer c_h^* so that if* $\mathbf{P}$ *is a poset of height h,* $\mathbf{P}$ *satisfies the conditions of the special case, and the cover graph of* $\mathbf{P}$ *is planar, then* $\dim(\mathbf{P}) \leq c_h^*$.

Proof. Combining Proposition 2.8.3, Corollary 2.9.4, Corollary 2.10.13, Corollary 2.10.15, and Proposition 2.5.3, we obtain:

$$c_h^* \leq 2h^3 R_2(3, h^2) R_3(h+1, 4)^2 \left(1 + 2^{2h+1} h R_2(h+1, 1; 3, 2h^2)\right) + 2.$$

In particular, c_h^* exists. □

2.12 The general case

In general, $\mathbf{G}$ may be disconnected. Label the components C_1, C_2, $\ldots$, C_m and for now consider only C_1. Let $a_0 \in \min(\mathbf{P})$ be any minimal element in C_1 and partition the minimal and maximal elements of $\mathbf{P}$ in C_1 according to the following definition:

- $A_0 = \{a_0\}$,
- for $i \geq 0$, $B_i = \{b \in \max(\mathbf{P}) \mid b \notin B_j$ for any $j < i$ and there is an $a \in A_i$ such that $a < b$ in $\mathbf{P}\}$,
- for $i \geq 1$, $A_i = \{a \in \min(\mathbf{P}) \mid a \notin A_j$ for any $j < i$ and there is a $b \in B_{i-1}$ such that $a < b$ in $\mathbf{P}\}$.

Since C_1 is connected we have indeed defined a partition of the minimal and maximal elements. Furthermore, the sequence $A_0, B_0, A_1, B_1, \ldots$ never has an empty set followed by a nonempty set. Also, since $\mathbf{P}$ is finite, there are only a finite number of nonempty sets.

Next define G_i to be the graph induced by A_i, B_i, and all vertices v such that $a < v < b$ in $\mathbf{P}$ for some $a \in A_i$ and $b \in B_i$. Similarly, define G_i' to be the graph induced by A_i, B_{i-1}, and all vertices v such that $a < v < b$ in $\mathbf{P}$ for some $a \in A_i$

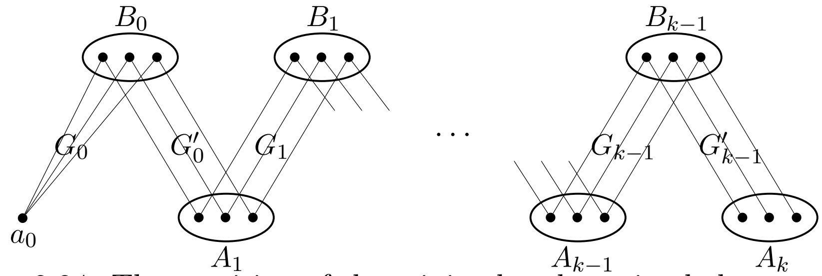


Figure 2.24: The partition of the minimal and maximal elements of C_1

and $b \in B_{i-1}$. See Figure 2.24. Notice that $G_i \cap G_j$ is empty for all $i \neq j$, $G_i' \cap G_j'$ is empty for all $i \neq j$, and $G_i \cap G_j'$ is empty unless j is i or $i-1$.

For $i \geq 1$ let X_i be the set of critical pairs $(a, b) \in \text{Crit}^*(\mathbf{P})$ with $a \in A_i$ and $b \in B_i$. The next lemma is the heart of the reduction from the general case to the special case.

Lemma 2.12.1. *Fix an $i \geq 1$ with X_i nonempty. The critical pairs in X_i can be reversed with c_h^* linear extensions.*

Proof. Let $\mathcal{G}$ be the graph induced by the vertices in G_0, G_0', …, G_{i-2}', G_{i-1}. Notice that $\mathcal{G}$ is connected, as all vertices have a path to a_0 in $\mathcal{G}$. Moreover, as noted above, $\mathcal{G}$ is disjoint from the graph induced by the vertices in $\cup_{j \geq i} G_j$ and $\cup_{j \geq i} G_j'$. Therefore, we may perform a graph-theoretic contraction of $\mathcal{G}$ to a single vertex $v_{\mathcal{G}}$ without affecting any of the subgraphs in $\cup_{j \geq i} G_j$ and $\cup_{j \geq i} G_j'$.

Consider the subgraph of C_1 induced by the vertices in G_0, G_0', …, G_i. Let $\mathcal{H}$ be the minor of this graph that is obtained when $\mathcal{G}$ is contracted to the vertex $v_{\mathcal{G}}$. Define $\mathbf{P}_{\mathcal{H}}$ to be the poset that results from these minor operations; that is, $\mathbf{P}_{\mathcal{H}}$ is the poset that satisfies:

- the cover graph of $\mathbf{P}_{\mathcal{H}}$ is $\mathcal{H}$,
- if x and y are vertices in $\mathcal{H} - v_{\mathcal{G}}$, then $x \leq y$ in $\mathbf{P}_{\mathcal{H}}$ if and only if $x \leq y$ in $\mathbf{P}$, and
- if x is a vertex in $\mathcal{H}$ that is adjacent to $v_{\mathcal{G}}$, then $x \leq v_{\mathcal{G}}$.

The subgraph of $\mathcal{H}$ induced by the vertices in G_i has remained exactly as it was in $\mathbf{G}$. Therefore, the subposet of $\mathbf{P}_\mathcal{H}$ when restricted to the elements of G_i is exactly the same as the subposet of $\mathbf{P}$ restricted to the same elements.

Notice that $v_\mathcal{G}$ is a maximal element of $\mathbf{P}_\mathcal{H}$. In fact, $v_\mathcal{G}$ is greater in $\mathbf{P}_\mathcal{H}$ than every element of A_i, and hence $v_\mathcal{G} > a$ for all minimal elements $a \in \mathbf{P}_\mathcal{H}$. Furthermore, the height of $\mathbf{P}_\mathcal{H}$ is at most the height of $\mathbf{P}$, and because the class of planar graphs is closed under taking minors, $\mathcal{H}$ is planar. Therefore, $\mathbf{P}_\mathcal{H}^D$, the dual of $\mathbf{P}_\mathcal{H}$, satisfies the conditions of the special case. By Theorem 2.11.1, we can reverse all critical pairs in $\mathbf{P}_\mathcal{H}^D$ with c_h^* linear extensions. Since (b, a) is a min-max critical pair in $\mathbf{P}_\mathcal{H}^D$ if and only if $(a, b) \in X_i$, we obtain the desired result. □

For $i \geq 1$, let Y_i be the set of critical pairs $(a, b) \in \mathrm{Crit}^*(\mathbf{P})$ with $a \in A_i$ and $b \in B_{i-1}$. The next lemma states that the bound on X_i applies to Y_i as well.

Lemma 2.12.2. *Fix an $i \geq 1$ with Y_i nonempty. The critical pairs in Y_i can be reversed with c_h^* linear extensions.*

Proof. Let $\mathcal{G}$ be the graph induced by the vertices in G_0, G'_0, …, G'_{i-2} and consider the subgraph of C_1 induced by G_0, G'_0, …, G'_{i-1}. The proof then follows analogously to the proof of Lemma 2.12.1: we contract $\mathcal{G}$ to a single vertex $v_\mathcal{G}$ to obtain the minor $\mathcal{H}$, and then we then define the poset $\mathbf{P}_\mathcal{H}$ (here $v_\mathcal{G}$ is a minimal element). In this case, $\mathbf{P}_\mathcal{H}$ satisfies the conditions of the special case, as opposed to the dual of $\mathbf{P}_\mathcal{H}$ in the previous case. We then apply Theorem 2.11.1 to obtain the desired result. □

The next lemma bounds the number of extensions needed to reverse all critical pairs that appear in some X_i or Y_i (that is, those critical pairs that have a signature) as a function of h.

Lemma 2.12.3. *The critical pairs in $(\bigcup_i X_i) \cup (\bigcup_i Y_i)$ can be reversed in $4c_h^*$ linear extensions of* $\mathbf{P}$.

Proof. Consider the set of critical pairs in $X_{\text{odd}} = \bigcup_i X_i$ for all odd indices i. We claim these can be reversed in c_h^* extensions, the same number required to reverse the pairs in any one such X_i. For this it suffices to show that, given any odd j and j' where $j \neq j'$, and any $(a, b) \in X_j$ and $(a', b') \in X_j'$, there is no alternating cycle amongst the critical pairs in X_{odd} containing both of these pairs (so, in particular, this holds when $\Sigma(a, b) = \Sigma(a', b')$). Without loss of generality, $j' \geq j + 2$.

Suppose that such an alternating cycle exists, with $a_{i-1} < b_i$ cyclically. Let $a' = a_i$ and $b' = b_i$ in the cycle. So $a' < b_{i+1}$ in $\mathbf{P}$. By construction, b_{i+1} is in some B_k for $k \geq j' - 1$; otherwise, a' would be in a set of smaller index. In fact, $k \geq j'$ since k must be odd as well for b_{i+1} to be in the alternating cycle. The same is true for a_{i+1} since it is in A_k. Repeating this argument cyclically we find that every element of $\mathbf{P}$ in the alternating cycle appears in a set of the partition whose index at least j', contradicting the assumption that the cycle contains (a, b).

Define $X_{\text{even}} = \bigcup_i X_i$ for all even indices i, define $Y_{\text{odd}} = \bigcup_i Y_i$ for all odd indices i, and define $Y_{\text{even}} = \bigcup_i Y_i$ for all even indices i. The analogous argument to the one above holds in each case. Therefore these four sets of critical pairs can each be reversed with c_h^* linear extensions. $\square$

We now turn our attention to all remaining critical pairs of C_1 in $\text{Crit}^*(\mathbf{P})$. Any such (a, b) has $a \in A_i$ and $b \in B_j$ with $|i - j| \geq 2$. Let S_1 be the subset of these critical pairs with $i < j$ and let S_2 be the subset with $i > j$.

Lemma 2.12.4. *The critical pairs in S_i can be reversed with one linear extension of $\mathbf{P}$, for each $i \in \{1, 2\}$.*

Proof. Suppose there is an alternating cycle with $a_{i-1} < b_i$, cyclically, amongst the critical pairs in S_1. Let m be the integer such that $a_i \in A_m$. Then $b_i \in B_n$ for some $n \geq m + 2$. By definition of the partition, the set containing a_{i-1} has an index in $\{n - 1, n, n + 1\}$. Therefore, the set that contains b_{i-1} has index at least $m + 3$.

Repeating this argument cyclically we find that b_{i+1} must belong to a set with index strictly bigger than $m + 1$. But, since $a_i < b_{i+1}$ in $\mathbf{P}$, we have contradicted the construction of the partition. So the alternating cycle does not exist. The analogous argument shows that S_2 does not contain an alternating cycle either. $\square$

Clearly, Lemmas 2.12.3 and 2.12.4 could have been applied to any component of $\mathbf{G}$. It is also clear that any alternating cycle in the set of critical pairs (a, b) for which a and b belong to the same component of $\mathbf{G}$ occurs in a single component, as any two elements in different components are incomparable in $\mathbf{P}$. Thus we have the following statement, which we state as a theorem for emphasis.

Theorem 2.12.5. *Consider the set of critical pairs (a, b) in $\mathrm{Crit}^*(\mathbf{P})$ such that a and b belong to the same component of the cover graph of $\mathbf{P}$. This set can be reversed with $4c_h^* + 2$ linear extensions of $\mathbf{P}$.*

The critical pairs (a, b) in $\mathrm{Crit}^*(\mathbf{P})$ that we have yet to deal with are those in which $a \in C_i$ and $b \in C_j$ for $i \neq j$; those whose coordinates come from different components of $\mathbf{G}$. Let S_1 be the subset of these critical pairs with $i < j$ and let S_2 be the subset with $i > j$.

Lemma 2.12.6. *The critical pairs in S_i can be reversed with one linear extension of $\mathbf{P}$, for each $i \in \{1, 2\}$.*

Proof. Suppose there is an alternating cycle with $a_{i-1} < b_i$, cyclically, amongst the critical pairs in S_1. Let m be the integer such that $a_i \in C_m$. Then $b_i \in C_n$ for some $n > m$. Then a_{i-1} is in C_n as well since there is a directed path in $\mathbf{G}$ from a_{i-1} to b_i. Thus the component containing b_{i-1} has index strictly greater than m as well. Repeating this argument cyclically yields a contradiction, since on one hand, the component containing b_{i+1} must have index strictly greater than m, and on the other hand, it must be in C_m for $a_i < b_{i+1}$ in $\mathbf{P}$. So, the alternating cycle cannot exist. An analogous argument shows that S_2 cannot contain an alternating cycle either. $\square$

Now we are prepared to finish the proof of the main theorem.

Theorem 2.12.7. *For every $h \geq 1$, there exists a least positive integer c_h so that if $\mathbf{P}$ is a poset of height h and the cover graph of $\mathbf{P}$ is planar, then* $\dim(\mathbf{P}) \leq c_h$.

Proof. As noted previously, $c_1 = 2$ and $c_2 = 4$, and so we may assume $h \geq 3$. By Lemma 2.4.5, c_h is bounded by the number of linear extensions required to reverse all critical pairs in $\mathrm{Crit}^*(\mathbf{P})$. Combining Theorem 2.12.5 and Lemma 2.12.6, we find

$$c_h \leq 4c_h^* + 4.$$

In particular, c_h exists. $\square$

2.13 The lower bound

Consider the poset whose cover graph is depicted in Figure 2.25. Just like Figure 2.7 from Section 2.3, this poset contains S_8 as a subposet and has height is 7. Thus $c_7 \geq 8$. This is just one example of an infinite family that demonstrates $c_h \geq h + 1$. In fact, we can do slightly better. Add an element a_9 to the infinite face of the embedding in Figure 2.25 and oriented edges (a_9, b_i) for $i \in [8]$. Then add an element b_9 in the face incident with $a_1, a_2, \ldots, a_8$ and oriented edges (a_i, b_9) for $i \in [8]$. It is easy to see that we can perform these additions in such a way as to maintain the planarity of the embedding. The new poset contains S_9 as a subposet and still has height 7. As before, we can extend this example to an infinite family that demonstrates $c_h \geq h+2$.

So, the best known lower bound on c_h is linear in h and the best known upper bound on c_h is enormous. We suspect that the c_h is much closer to the lower bound than the upper bound. In fact, we have no reason to believe that $h + 2$ is not the correct answer. A super-linear lower bound on c_h would require a markedly different construction.

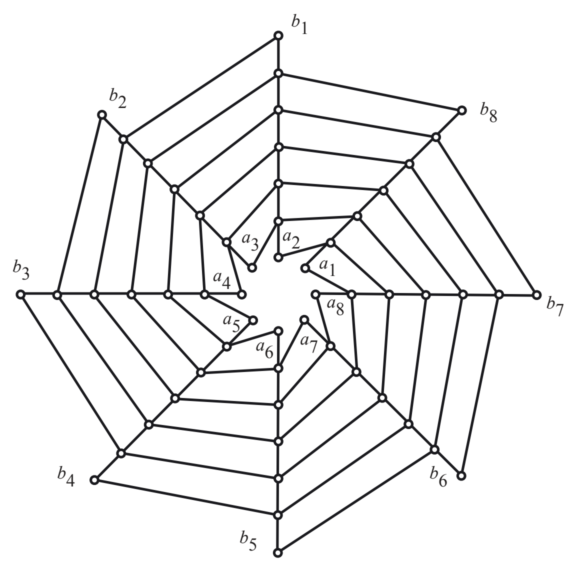


Figure 2.25: A poset with a planar cover graph and S_8 as a subposet

CHAPTER III

HAMILTONIAN CYCLES IN SUBSET LATTICES

3.1 Introduction

For a positive integer n, we let $\mathcal{B}(n)$ denote the subset lattice consisting of all subsets of $[n]$ ordered by inclusion. Of course, we may also consider $\mathcal{B}(n)$ as the set of all 0–1 strings of length n with partial order

$$\mathbf{a} = (a_1, a_2, \dots, a_n) \le \mathbf{b} = (b_1, b_2, \dots, b_n)$$

if and only if $a_i \le b_i$ for each $i = 1, 2, \dots, n$. We illustrate this with a diagram for $\mathcal{B}(4)$ in Figure 3.1.

Some elementary properties of the poset $\mathcal{B}(n)$ are:

(1) The height is $n+1$ and all maximal chains have exactly $n+1$ points.

(2) The size of the poset $\mathcal{B}(n)$ is 2^n and the elements are partitioned into ranks (antichains) $A_0, A_1, \dots, A_n$ with $|A_i| = \binom{n}{i}$ for each $i = 0, 1, \dots, n$.

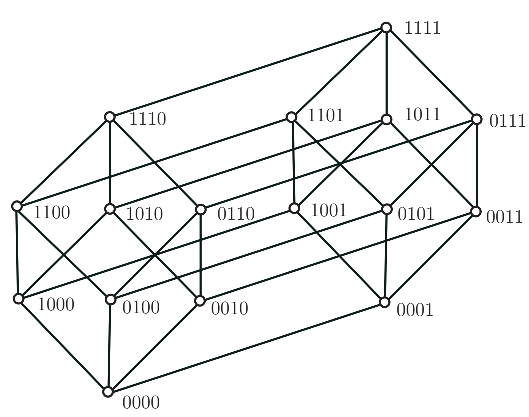


Figure 3.1: The subset lattice $\mathcal{B}(4)$

(3) The maximum size of a rank in the subset lattice occurs in the middle, i.e. if $s = \lfloor n/2 \rfloor$, then the largest binomial coefficient in the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ is $\binom{n}{s}$. Note that when n is odd, there are two ranks of maximum size, but when n is even, there is only one.

For the width of the subset lattice, we have the following classic result due to Sperner [46].

Theorem 3.1.1 (Sperner). *For each $n \geq 1$, the width of the subset lattice $\mathcal{B}(n)$ is the maximum size of a rank, i.e.,*

$$\mathrm{width}(\mathcal{B}(n)) = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

There have been a number of elegant proofs for Sperner's theorem, including the independent but roughly equivalent arguments taken by Lubell [36] (whose result is a special case of an earlier lemma of Bollobás [9]), Yamamoto [55], and Meshalkin [37]. A second approach was initiated by de Bruijn *et al* [13], Katona [30], and Kleitman [33] using symmetric chains. (This approach was later used as a method of attack on the well-known Littlewood-Offord problem [35]. Erdős [16] noticed that Sperner's theorem implies the best bound for the real-number version of Littlewood-Offord. Later, Kleitman [34] used symmetric chains to solve the full Littlewood-Offord problem.) In light of Dilworth's theorem it is no surprise that one can prove Sperner's result by partitioning $\mathcal{B}(n)$ into the appropriate number of chains. We present this approach next since we use it as motivation for our results.

3.1.1 Proving Sperner with symmetric chains

A poset $\mathbf{P}$ is said to be *ranked* if all maximal chains have the same cardinality. When a poset is ranked, then there is a partition $X = A_1 \cup A_2 \cup \ldots A_h$ so that every maximal chain consists of exactly one point from each A_i. We call this partition its *partition into ranks.*

A ranked poset is said to be *Sperner* if the width of the poset is just the maximum cardinality of a rank. So using this terminology, Sperner's theorem is just the assertion that the subset lattice is Sperner.

Let $\mathbf{P}$ be a ranked poset of height h and let A_1, $A_2, \ldots, A_h$ be the ranks of $\mathbf{P}$. A chain C in $\mathbf{P}$ is called a *symmetric chain* if there exists an integer s so that C contains exactly one point from each rank $A_s, A_{s+1}, \ldots, A_{h+1-s}$. Intuitively, a symmetric chain is (1) balanced about the middle of the poset and (2) dense in the sense that it is not possible to insert a point in between two consecutive points in C.

The following proposition is self-evident.

Proposition 3.1.2. *If a ranked poset has a partition into symmetric chains, then it is a Sperner poset. In fact, its width is just the size of the middle rank(s).*

So an alternative proof of Sperner's theorem is provided by the following result, due independently to de Bruijn *et al* [13], Katona [30], and Kleitman [33].

Theorem 3.1.3. *For each $n \geq 1$, the subset lattice $\mathcal{B}(n)$ has a symmetric chain partition.*

In fact, a stronger result can be established. But first, we need a definition. Let $\mathbf{P} = (X, P)$ and $\mathbf{Q} = (Y, Q)$ be posets. The *cartesian product* $\mathbf{P} \times \mathbf{Q}$ is the poset with ground set $X_P \times X_Q$ and partial order $\{((a_1, b_1), (a_2, b_2)) \mid (a_1, a_2) \in P \text{ and } (b_1, b_2) \in Q\}$. We can now state the stronger result.

Theorem 3.1.4. *If $\mathbf{P}$ and $\mathbf{Q}$ are ranked posets and each has a symmetric chain partition, then $\mathbf{P} \times \mathbf{Q}$ is ranked and has a symmetric chain partition.*

Note that Theorem 3.1.3 follows immediately from Theorem 3.1.4 since $\mathcal{B}(n)$ is just the cartesian product of n copies of the two-element chain $\mathbf{2}$, and this has a trivial symmetric chain partition.

The argument for Theorem 3.1.4 begins with a technical lemma.

Lemma 3.1.5. *Let m and n be positive integers. Then the cartesian product $\mathbf{m} \times \mathbf{n}$ has a symmetric chain partition.*

Proof. The point set of $\mathbf{m} \times \mathbf{n}$ is just $\{(i, j) : 0 \le i < m, 0 \le j < n\}$. Without loss of generality $m \le n$, so that the width of $\mathbf{m} \times \mathbf{n}$ is m. Then for each $i = 0, 1, \ldots, m-1$, let

$$C_i = \{(i, 0), (i, 1), \ldots, (i, n-1-i), (i+1, n-1-i), \ldots, (m-1, n-1-i)\}.$$

Then the family $\{C_1, C_2, \ldots, C_m\}$ is a symmetric chain partition of $\mathbf{m} \times \mathbf{n}$. $\square$

We are now ready for the proof of Theorem 3.1.4. It is easy to see that if (1) $\mathbf{P}$ is ranked and has height h_1, and (2) $\mathbf{Q}$ is ranked and has height h_2, then $\mathbf{P} \times \mathbf{Q}$ is ranked and has height $h_1 + h_2 - 1$. Now suppose that $\mathbf{P}$ and $\mathbf{Q}$ have symmetric chain partitions. Let C be a chain from the partition of $\mathbf{P}$ and let D be a chain from the partition of $\mathbf{Q}$. Then apply Lemma 3.1.5 to obtain a partition of the product $C \times D$. What results is a symmetric chain partition of $\mathbf{P} \times \mathbf{Q}$.

The remainder of the chapter is organized as follows. In Section 3.2, we define a class of posets that generalizes ranked posets; namely, leveled posets. In Section 3.3 we define the HC-SCP property and the strong HC-SCP property, the primary definitions in this work. In Section 3.4 we prove that $\mathcal{B}(n)$ has the strong HC-SCP property, and in Section 3.5 we prove that the strong HC-SCP property is weakly closed under cartesian product. Finally, in Section 3.7 we connect our results to open problems in this area.

3.2 Leveled posets

A slightly more general class of posets than ranked posets is leveled posets. A poset $\mathbf{P}$ is *leveled* if there is a partition $\mathbf{P} = A_1 \cup A_2 \cup \cdots \cup A_h$, where h is the height of $\mathbf{P}$, with each A_i an antichain, so that if x covers y in $\mathbf{P}$, there is some $i \ge 2$ for which $x \in A_i$ and $y \in A_{i-1}$. Naturally, we refer to the antichains as levels, and note that

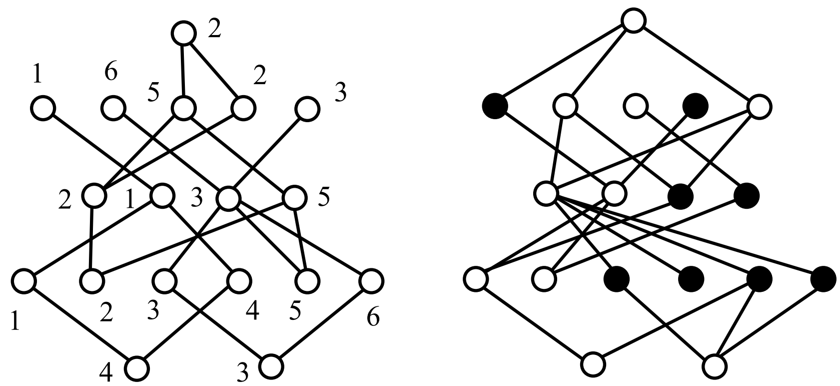


Figure 3.2: Two leveled posets — only one is Sperner

the width of $\mathbf{P}$ is at least as large as the maximum size of a level. Ranked posets, and in particular the subset lattices, are leveled. However, not all leveled posets are ranked. When $\mathbf{P}$ is a connected leveled poset the antichain partition is unique, and in the treatment to follow, we will only consider connected posets.

The following two definitions are similar to those presented in the previous section for ranked posets, and we choose to state them again for clarity. A leveled poset is called a *Sperner* poset when its width is equal to the maximum size of a level. In Figure 3.2, we show two leveled posets. In both, the sizes of the levels are 1, 2, 4, 5 and 6. The poset on the left is Sperner, and the numbers on the figure indicate a partition into six chains. The poset on the right is not a Sperner poset, as the darkened points form an antichain of size 8.

Let $\mathbf{P}$ be a leveled poset of height h. A chain $C = \{x_1 < x_2 < \cdots < x_m\}$ is *symmetric* if (1) $x_1 \in A_i$ implies that $x_m \in A_{h+1-i}$, and (2) x_{i+1} covers x_i for each $i = 1, 2, \ldots, m-1$. A chain partition of $\mathbf{P}$ is called a *symmetric chain partition* when each chain in the partition is a symmetric chain in $\mathbf{P}$. The following proposition is self-evident.

Proposition 3.2.1. *Let* $\mathbf{P}$ *be a leveled poset. If* $\mathbf{P}$ *has a symmetric chain partition, then* $\mathbf{P}$ *is a Sperner poset. Furthermore, if* h *is the height of* $\mathbf{P}$*, then* $|R_i| = |R_{h+1-i}|$,

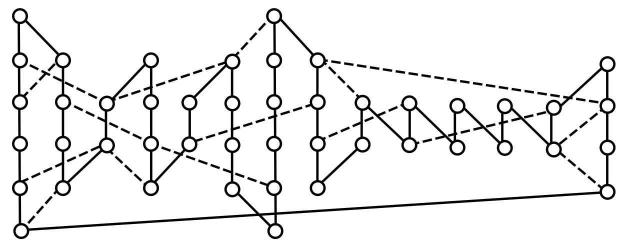


Figure 3.3: Symmetric chain partition of a leveled poset

for every $i = 1, 2, \ldots, \lfloor h/2 \rfloor$*. Moreover, if* $h = 2r+1$*, then the width of* $\mathbf{P}$ *is* $|R_{r+1}|$*, and if* $h = 2r$*, then the width of* $\mathbf{P}$ *is* $|R_r| = |R_{r+1}|$*.*

In Figure 3.3, we show a leveled poset, with some of the covers shown as dashed lines (we will explain this detail shortly), and the points have been arranged so that the vertical chains form a symmetric chain partition.

Recall that the fact that $\mathcal{B}(n)$ has a symmetric chain partition was proved in two steps — Lemma 3.1.5 followed by Theorem 3.1.4. The reader should look for similar steps as we proceed.

3.3 Symmetric chain partitions and hamiltonian cycles

The cover graph $Q(n)$ of the subset lattice $\mathcal{B}(n)$ is called a *cube*. These graphs have been studied extensively, as they exhibit many interesting combinatorial properties. Here is a combinatorial gem, a result frequently used in elementary combinatorics and graph theory classes to illustrate the power of induction.

Theorem 3.3.1. *For* $n \geq 2$*, the cube* $Q(n)$ *is hamiltonian.*

Proof. Evidently, $Q(2)$ is hamiltonian, as evidenced by the sequence: $((0,0), (0,1), (1,1), (1,0))$. Assume that $Q(k)$ is hamiltonian and list the cycle as $(A_1, A_2, \ldots, A_t)$, where $t = 2^n$. Then

$$(A_1, A_2, \ldots, A_{t-1}, A_t, A_t \cup \{k+1\}, A_{t-1} \cup \{k+1\}, \ldots, A_2 \cup \{k+1\}, A_1 \cup \{k+1\})$$

is a hamiltonian cycle in $Q(k+1)$. $\square$

3.3.1 Hamiltonian Cycle–Symmetric Chain Partition property

Take a second look at Figure 3.3 and observe that the covers which are shown by solid lines form a hamiltonian cycle in the cover graph of $\mathbf{P}$. We say that a leveled poset has the Hamiltonian Cycle–Symmetric Chain Partition property, which we abbreviate as the HC-SCP property, if its cover graph has a hamiltonian cycle which parses into a symmetric chain partition. Eventually, we will show that the subset lattice has the HC-SCP property, but we elect to obtain this result as a special case of a more general treatment, just as de Bruijn, Katona and Kleitman did for the symmetric chain property.

3.3.2 The special role of a 2-element chain

By convention, we say that a connected graph on two vertices has a hamiltonian cycle in the sense that all vertices can be listed so that (1) no vertex appears twice in the list, (2) consecutive vertices are adjacent, and (3) the last vertex is also adjacent to the first. So with this convention, we could say that the cube $Q(n)$ is hamiltonian for all $n \geq 1$. In the same sense, we want to develop a framework for studying leveled posets with the HC-SCP property so that if $\mathbf{P}$ has this property, so does the cartesian product $\mathbf{P} \times \mathbf{2}$. But there are challenges to achieving this goal.

Example 3.3.2. Consider the leveled poset $\mathbf{P}$ shown in Figure 3.4. Evidently $\mathbf{P}$ has the HC-SCP property. However, the cartesian product $\mathbf{P} \times \mathbf{2}$ does not. To see this, we need a bit of case analysis. Let $\mathbf{2} = (0, 1)$. To be concise, we will refer to $(x, 0)$ as x (the label x comes from Figure 3.4) and to $(x, 1)$ as x'.

By way of contradiction, suppose that $\mathbf{P} \times \mathbf{2}$ has a hamiltonian cycle H that parses into a symmetric chain decomposition with chains $C_1, C_2, \ldots, C_w$, where w is the width of $\mathbf{P} \times \mathbf{2}$. Notice that all chains in H must have odd height since $\mathbf{P} \times \mathbf{2}$ has height seven. Without loss of generality we may assume that a_1 is the starting point of H.

Suppose first C_1 is $a_1a_2a_3a_4a_5a_6a'_6$. Then C_2 has length five and proceeds downward. If C_2 starts with b'_5, then the only way H can reach a'_5 is with the chain $a'_1a'_2a'_3a'_4a'_5$, at which point H has no way of leaving a'_5. Thus C_2 must be $a'_5a'_4a'_3a'_2a'_1$ and C_3 must be $d'_2d'_3d'_4$. If C_4 is $d'_5d_5d_4d_3d_2$, then all neighbors of a_1 are used before H visits all points in the poset, a contradiction. So C_4 must be the singleton d_4. However, there are now no valid options for C_5, a contradiction.

Now suppose C_1 is $a_1a'_1a'_2a'_3a'_4a'_5a'_6$. If C_2 is $a_6a_5a_4a_3a_2$, then all neighbors of a_2 have been used and H is stuck with no choice for C_3. So C_2 starts with b'_5 and ends at either b_2 or c'_1. If its the former, then C_2 must be $b'_5b'_4b'_3b'_2b_2$ in order to be able to get to a_2. But even in this case, the only way to get to a_2 is for C_3 to be $b_3b_4b_5$ and C_4 to be $a_6a_5a_4a_3a_2$, in which case H is stuck. If its the latter, then C_2 is $b'_5b'_4b'_3b'_2c'_1$. But then the only way to use b_2 is in some chain C_i that is $b_2b_3b_4b_5a_6$. Then C_{i+1} is $a_5a_4a_3$, and there is no way to use a_2, a contradiction.

If C_1 is any other chain that ends with a'_6, then there is some $j \in \{2, 3, 4, 5\}$ such that $a_j \in C_1$ but $a_{j+1} \notin C_1$. If $j \leq 4$, then a_{j+1} has a unique neighbor not in C_1, namely a_{j+2}. So H must use a_{j+2} just prior to a_{j+1} in some chain C_i, but then H is stuck. If $j = 5$, then we make a similar case for a'_4; the only way to use a'_4 is with a chain that uses a'_3 just prior, but then H is again stuck.

Thus C_1 ends at c'_6. Suppose C_1 is $a_1a'_1d'_2d'_3d'_4d'_5c'_6$. If C_2 is $c_6d_5d_4d_3d_2$, then there is no choice for C_3. If C_2 ends at c_2, then the only way for H to be able to reach d_j for $j \in \{2, 3, 4, 5\}$ is for C_2 to be $c'_5c'_4c'_3c'_2c_2$. Then C_3 is $c_3c_4c_5$ and C_4 is $c_6d_5d_4d_3d_2$, and H is then stuck. So C_2 is $c'_5c'_4c'_3c'_2c'_1$. But then the only chain that could contain c_6 is some C_i that is $c_2c_3c_4c_5c_6$. Then C_{i+1} must be $d_5d_4d_3$, and there is no valid way to use d_2.

So we may assume that C_1 includes d_2. If C_1 is $a_1d_2d_3d_4d_5c_6c'_6$, then C_2 must be $d'_5d'_4d'_3d'_2a'_1$; otherwise H cannot hit d'_5. But then C_3 is $a'_2a'_3a'_4$ and C_4 is $a'_5a_5a_4a_3a_2$, and H is stuck. Thus C_1 does not contain c_6. So there is some $j \in \{2, 3, 4, 5\}$ such

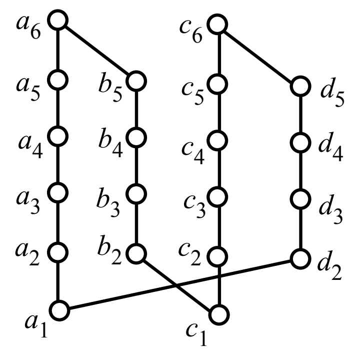


Figure 3.4: A troublesome poset

that $d_j, d'_j \in C_1$. If $j \geq 3$, then there is no valid way for H to contain d'_{j-1} (if some chain did, then H would be stuck). Similarly, if $j = 2$, then no chain of H can contain d_3 without getting stuck. Therefore $\mathbf{P} \times \mathbf{2}$ does not have the HC-SCP property.

3.3.3 A stronger property

Let $\mathbf{P}$ be a leveled poset and let h and w denote respectively, the height and width of $\mathbf{P}$. We say that $\mathbf{P}$ satisfies the *strong* HC-SCP property when there is a hamiltonian cycle H in the cover graph of $\mathbf{P}$ so that (1) H parses into a symmetric chain partition consisting of the chains $C_1, C_2, \ldots, C_w$ labeled in the order they are encountered in traveling around H and (2) the chains in this partition can be partitioned into non-empty blocks $B_1, B_2, \ldots, B_s$, with all chains in a block occurring consecutively (in the cyclic sense) in H, so that for each $i = 1, 2, \ldots, s$, one of the following statements applies:

(1) $|B_i| = 2$, and if $B_i = \{C, C'\}$ with $|C'| = 2 + |C|$, then the least element of C covers the least element of C' and the greatest element of C is covered by the greatest element of C'.

(2) h is even and B_i consists of a single 2-element chain.

We refer to blocks as Type 1 or Type 2 according to whether the first or the second of these two statements holds. When h is odd, note that any poset satisfying

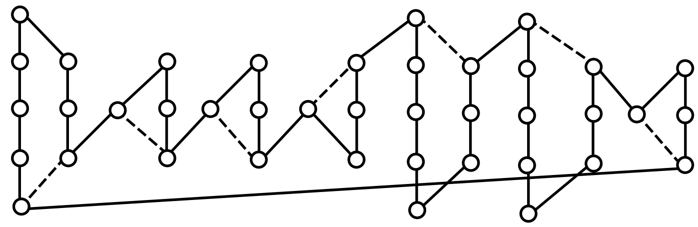


Figure 3.5

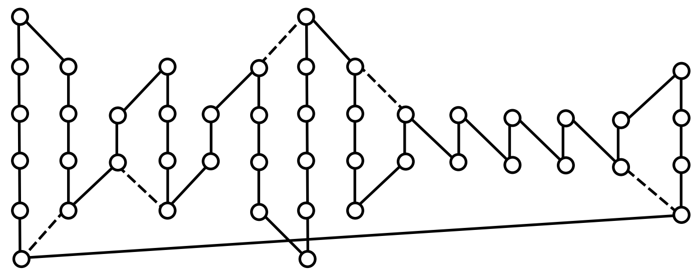


Figure 3.6

the HC-SCP property has even width. Furthermore, only Type 1 blocks can be used. We show in Figure 3.5 a leveled poset of height five satisfying the strong HC-SCP property. When h is even, a poset satisfying the HC-SCP property can have even width or odd width. In Figure 3.6, we show a leveled poset of height six which satisfies the strong HC-SCP property. Here there are four Type 2 blocks. Note that not all 2-element chains form Type 2 blocks. Some of them may be absorbed in Type 1 blocks with the other chain having four elements.

3.4 The strong theorem

We are now positioned to prove the following structural theorem.

Theorem 3.4.1. *Let* $\mathbf{P}$ *be a leveled poset. If* $\mathbf{P}$ *satisfies the strong HC-SCP property, so does the cartesian product* $\mathbf{P} \times \mathbf{2}$.

Proof. We start with a hamiltonian cycle H in $\mathbf{P}$ that parses into symmetric chains, with $B_1, B_2, \ldots, B_s$ the partition witnessing that $\mathbf{P}$ satisfies the strong HC-SCP property. We will then proceed to construct the required hamiltonian cycle H' in $\mathbf{P} \times \mathbf{2}$, together with the specification of the blocks which show that $\mathbf{P} \times \mathbf{2}$ also satisfies the strong HC-SCP property. We find it useful to use the following natural notation and terminology. A set of points which occur consecutively in a hamiltonian cycle will be called a *group*. Abusing notation slightly, we will also consider each block B_i as a group in H, so we will talk about H entering the block B_i at a point $x \in C$ from B_i and leaving it at a point $y \in C'$ from B_i. Note that when B_i is a Type 2 block, the points x and y are just the top and bottom points of the same chain.

Our construction for H' will satisfy the following properties.

(1) For each $i = 1, 2, \ldots, s$, the elements of $G_i = \bigcup\{C \times \{0, 1\} : C \in B_i\}$ will be a group in H'.

(2) If H enters block B_i at $x \in C$ and exits B_i at $y \in C'$, then H' will enter G_i at $(x, 0)$ and it will exit G_i at $(y, 0)$.

(3) If H exits B_i at y and enters B_{i+1} at z, then H' leaves G_i at $(y, 0)$ and goes immediately to $(z, 0)$ in G_{i+1}.

Now here is how blocks of the two types will be handled.

(1) If B_i is a Type 1 block and the shorter chain has $r \geq 2$ elements, then G_i will consist of four chains in H' and they will be partitioned into two Type 1 blocks. The chain sizes will be $r + 3$ and $r + 1$ in one of them and $r + 1$ and $r - 1$ in the other.

(2) If B_i is a Type 1 block and the shorter chain has a single element, G_i will consist of three chains. Two of them will have sizes 4 and 2 and will form a Type 1 block in H'. The third chain will have size 2 and is thus a Type 2 block.

(3) If B_i is a Type 2 block, then G_i will consist of a Type 1 block, with one chain of size 3 and the other chain a singleton.

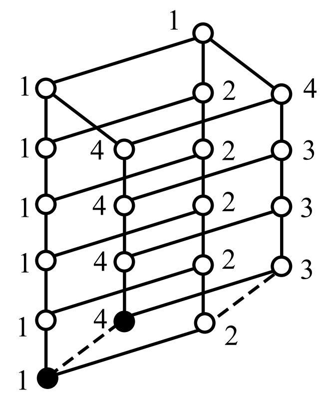


Figure 3.7: Large Type 1 blocks

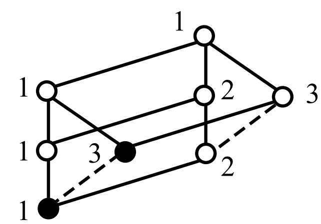


Figure 3.8: Small Type 1 blocks

With these specifications in mind, the remaining details of the construction can be verified by referring to three figures:

First, in Figure 3.7, we illustrate how H' will traverse the group G_i when B_i is a Type 1 block with the smaller chain containing at least two points. The darkened points represent the entering and exiting points. The illustration has them both on the bottom, but the picture can be inverted when they are on the top. Also, which of the points is the entering point and which is the exiting point can be reversed.

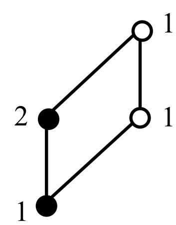


Figure 3.9: Type 2 blocks

Regardless, we see that the four chains in H' form two Type 1 blocks. The reader should note how the extra cover required in the definition of the *strong* HC-SCP is used to move from the second chain to the third in this construction.

Second, referring to Figure 3.8, we illustrate how H' will traverse the group G_i when B_i is a Type 1 block with the smaller chain being a singleton. Here we note that one of the 2-element chains is used with a 4-element chain in forming a Type 1 block, while the remaining 2-element chain forms a Type 2 block. As above, the extra cover is essential.

Finally, we note that the case of a Type 2 block is handled as depicted in Figure 3.9. Once the implications of the constructions detailed in these three figures has been digested, the proof of the theorem is complete. $\square$

Corollary 3.4.2. *For each $n \geq 1$, the subset lattice $\mathcal{B}(n)$ satisfies the strong HC-SCP property.*

Proof. The proof is a trivial induction, starting with the base case $n = 1$ where the hamiltonian cycle is a single Type 2 block. $\square$

3.4.1 Hamiltonian paths

We say that a leveled poset $\mathbf{P}$ satisfies the HP-SCP property if it has a hamiltonian path which parses into a symmetric chain partition. The *strong* HP-SCP property is then defined in an analogous manner. For example, when $m, p \geq 3$, the cartesian product $\mathbf{m} \times \mathbf{p}$ does not satisfy the HC-SCP property. Nevertheless, it does satisfy the strong HP-SCP property. The same argument used to show Theorem 3.4.1 also works for paths.

Corollary 3.4.3. *If $\mathbf{P}$ is a leveled poset satisfying the strong HP-SCP property, then so does the cartesian product $\mathbf{P} \times \mathbf{2}$.*

3.5 The strong property is weakly closed

In this section, we prove that the cartesian product of any two posets with the strong HC-SCP property has the HC-SCP property. The question of whether the cartesian product has the strong HC-SCP property remains open. We discuss this issue in greater detail in Section 3.6.

3.5.1 Hamiltonian paths in the product of chains

We start by identifying particular hamiltonian paths in the cartesian product of chains. Let m and p be positive integers. We say that the pair (m, p) is *Type H_1* if:

- $\mathbf{m} \times \mathbf{p}$ has a hamiltonian path H that parses into a symmetric chains, and
- H starts at $(0, 0)$ and ends at $(0, p-1)$.

Similarly, we say that the pair (m, p) is *Type H_2* if:

- $\mathbf{m} \times \mathbf{p}$ has a hamiltonian path H that parses into a symmetric chains, and
- H starts at $(0, 0)$ and ends at $(m-1, 0)$.

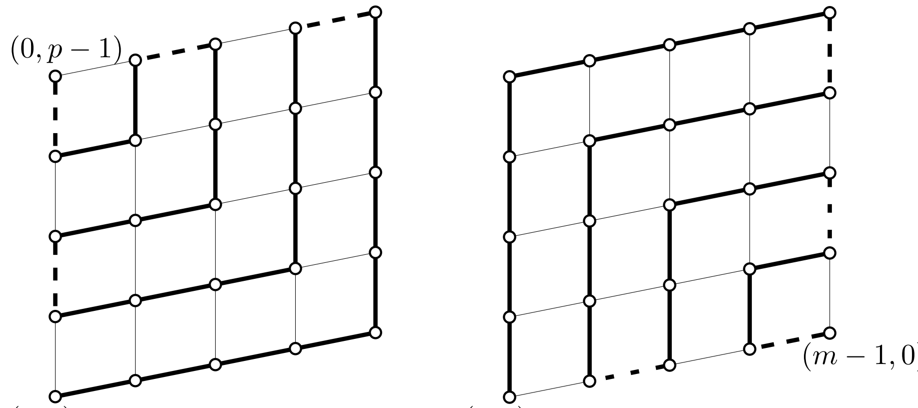
The following facts are easily verified.

Fact 3.5.1. If m is a positive integer, then $\mathbf{m} \times \mathbf{m}$ is both Type H_1 and Type H_2.

Fact 3.5.2. Let m and p be positive integers. Then $\mathbf{m} \times \mathbf{p}$ is Type H_1 if and only if $\mathbf{p} \times \mathbf{m}$ is Type H_2.

Together with Fact 3.5.1, the following two lemmas imply that (m, p) is Type H_1 or Type H_2 for all values of m and p.

Lemma 3.5.3. *Let m and p be positive integers. If $m < p$ and m is odd, then (m, p) is Type H_1. If $m > p$ and p is odd, then (m, p) is Type H_2.*



(a) $\mathbf{5} \times \mathbf{5}$ is Type H_1. (b) $\mathbf{5} \times \mathbf{5}$ is Type H_2.

Figure 3.10: Here $m = p = 5$. The union of the bold and dashed edges forms a hamiltonian path. The bold edges represent symmetric chains.

Proof. Assume $m < p$ with m odd. Consider the subposet $\mathbf{m} \times \mathbf{m}$ obtained by restricting the second coordinate to values at most $m-1$. By Fact 3.5.1, this subposet has a hamiltonian path that witnesses the fact that $\mathbf{m} \times \mathbf{m}$ is Type H_1. Call this path H, and let $C_1, C_2, \ldots, C_m$ be the symmetric chains that H parses into, in the order that they appear. We now extend H to the remainder of $\mathbf{m} \times \mathbf{p}$ in the following manner: to C_i add the elements $(m-i, m)$, $(m-i, m+1)$, $\ldots$, $(m-i, p-1)$. It is easily verified that this new hamiltonian path witnesses the fact that $\mathbf{m} \times \mathbf{p}$ is Type H_1.

Now assume $p < m$ and p is odd. Applying Fact 3.5.2 to the previous case we find that (m, p) is Type H_2. $\square$

Lemma 3.5.4. *Let m and p be positive integers. If $m < p$ and m is even, then (m, p) is Type H_2. If $m > p$ and p is even, then (m, p) is Type H_1.*

Proof. Assume $m < p$ and m is even. Consider the subposet $\mathbf{m} \times \mathbf{m}$ obtained by restricting the second coordinate to values at least $p-m$. By Fact 3.5.1, this subposet has a hamiltonian path that witnesses the fact that $\mathbf{m} \times \mathbf{m}$ is Type H_2. Call this path H, and let $C_1, C_2, \ldots, C_m$ be the symmetric chains that H parses into, in the order that they appear. We now extend H to the remainder of $\mathbf{m} \times \mathbf{p}$ in the following manner: to C_i add the elements $(i-1,0)$, $(i-1,1)$, ..., $(i-1,p-m-1)$. It is easily verified that this new hamiltonian path witnesses the fact that $\mathbf{m} \times \mathbf{p}$ is Type H_2.

Now assume $p < m$ and p is even. Applying Fact 3.5.2 to the previous case we find that (m,p) is Type H_1. $\square$

The table below summarizes Lemmas 3.5.3 and 3.5.4 for small values of m and p.

$\mathbf{m} \backslash \mathbf{p}$	1	2	3	4	5	6	7
1	-	H_1	H_1	H_1	H_1	H_1	H_1
2	H_2	-	H_2	H_2	H_2	H_2	H_2
3	H_2	H_1	-	H_1	H_1	H_1	H_1
4	H_2	H_1	H_2	-	H_2	H_2	H_2
5	H_2	H_1	H_2	H_1	-	H_1	H_1
6	H_2	H_1	H_2	H_1	H_2	-	H_2
7	H_2	H_1	H_2	H_1	H_2	H_1	-

The following corollary is now immediate.

Corollary 3.5.5. *If $m \geq 3$, then there exists a $\Gamma \in \{H_1, H_2\}$ such that both (m,p) and $(m-2,p)$ are Type Γ.*

3.5.2 Gluing hamiltonian paths together

Let C_1 and C_2 be chains with $|C_1| = m$ and $|C_2| = p$. Denote the (unique) least and greatest elements of $C_1 \times C_2$ by $S(C_1,C_2)$ and $\overline{S}(C_1,C_2)$, respectively. Further denote $(0,p-1)$ by $F_{\mathrm{H}_1}(C_1,C_2)$ and $(m-1,0)$ by $F_{\mathrm{H}_2}(C_1,C_2)$. Notice that, for $\Gamma \in \{\mathrm{H}_1,\mathrm{H}_2\}$, if $C_1 \times C_2$ is Type Γ, then it has a hamiltonian path from $S(C_1,C_2)$ to $F_\Gamma(C_1,C_2)$ that parses into symmetric chains.

Recall that a poset with the strong HC-SCP property has a hamiltonian cycle that can be partitioned into blocks. A Type 1 block consists of two chains C and C' with $|C'| = 2 + |C|$ with the least element of C covering the least element of C' and the greatest element of C covered by the greatest element of C'.

Let $B = \{C', C\}$ be a Type 1 block and let C'' be a chain. Denote by $\mathrm{HP}_{\mathrm{S}}(B, C'')$ a hamiltonian path in $B \times C''$ with the following properties:

- $\mathrm{HP_S}(B, C'')$ parses into symmetric chains,
- the ends of $\mathrm{HP_S}(B, C'')$ are $S(C', C'')$ and $S(C, C'')$, and
- the edge with ends $S(C', C'')$ and $S(C, C'')$ is not in $\mathrm{HP_S}(B, C'')$.

Similarly, denote by $\mathrm{HP_F}(B, C'')$ a hamiltonian path in $B \times C''$ with the following properties:

- $\mathrm{HP_F}(B, C'')$ parses into symmetric chains,
- the ends of $\mathrm{HP_F}(B, C'')$ are $F_{\mathrm{H_2}}(C', C'')$ and $F_{\mathrm{H_2}}(C, C'')$, and
- the edge with ends $F_{\mathrm{H_2}}(C', C'')$ and $F_{\mathrm{H_2}}(C, C'')$ is not in $\mathrm{HP_F}(B, C'')$.

Before we prove that $\mathrm{HP_S}(B, C'')$ and $\mathrm{HP_F}(B, C'')$ exist, we need to recall the following elementary definitions. The *dual of a partial order* P on a ground set X is denoted P^D and is defined by $P^D = \{(a, b) \mid (b, a) \in R\}$. The *dual* of a poset $\mathbf{P} = (X, P)$ is denoted $\mathbf{P}^D$ and is defined to have ground set X and partial order P^D. Intuitively, $\mathbf{P}^D$ is the upside-down version of $\mathbf{P}$.

Theorem 3.5.6. *Let $B = \{C', C\}$ be a Type 1 block and let C'' be a chain. Then $HP_S(B, C'')$ and $HP_F(B, C'')$ exist.*

Proof. Let $|C'| = m$ and $|C''| = p$. Clearly $m \geq 3$. By Corollary 3.5.5, there is a $\Gamma \in \{\mathrm{H_1}, \mathrm{H_2}\}$ such that both (m, p) and $(m-2, p)$ are Type Γ. Let H' and H be the Type Γ hamiltonian paths in $C' \times C''$ and $C \times C''$, respectively.

We find $\mathrm{HP_S}(B, C'')$ in the following manner: traverse H' from $S(C', C'')$ to $F_\Gamma(C', C'')$, follow the edge from $F_\Gamma(C', C'')$ to $F_\Gamma(C, C'')$, and traverse H from $F_\Gamma(C, C'')$ to $S(C, C'')$. For an example, see Figure 3.11.

If $\Gamma = \mathrm{H_2}$, then we find $\mathrm{HP_F}(B, C'')$ in the following manner: traverse H' from $F_{\mathrm{H_2}}(C', C'')$ to $S(C', C'')$, follow the edge from $S(C', C'')$ to $S(C, C'')$, then traverse H from $S(C, C'')$ to $F_{\mathrm{H_2}}(C, C'')$.

If $\Gamma = \mathrm{H}_1$, then $(C' \times C'')^D$, the dual of $C' \times C''$, is also Type H_1. The hamiltonian path that witnesses this is a hamiltonian path in $C' \times C''$ that parses into symmetric chains and has ends $\overline{S}(C', C'')$ and $F_{\mathrm{H}_2}(C', C'')$. Call this path $\overline{H}'$. Similarly we can find $\overline{H}$ in $C \times C''$ with ends $\overline{S}(C, C'')$ and $F_{\mathrm{H}_2}(C, C'')$. Now we find $\mathrm{HP}_{\mathrm{F}}(B, C'')$ in the following manner: traverse $\overline{H}'$ from $F_{\mathrm{H}_2}(C', C'')$ to $\overline{S}(C', C'')$, follow the edge from $\overline{S}(C', C'')$ to $\overline{S}(C, C'')$, then traverse H from $\overline{S}(C, C'')$ to $F_{\mathrm{H}_2}(C, C'')$. $\square$

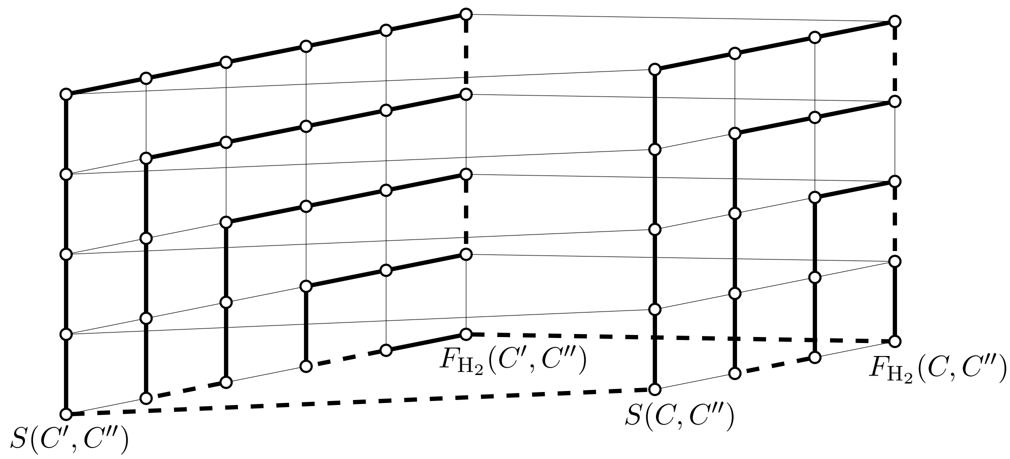


Figure 3.11: The union of the bold and dashed lines is $\mathrm{HC}_{\mathrm{S}}(B, C'')$, where $B = \{C', C\}$, $|C'| = 6$, and $|C''| = 5$; the bold lines form symmetric chains.

As a corollary to Theorem 3.5.6, it is clear that $\mathrm{HP}_{\mathrm{S}}(B, C'')$ and $\mathrm{HP}_{\mathrm{F}}(B, C'')$ exist in $(B \times C'')^D$. We denote these hamiltonian paths $\overline{\mathrm{HP}_{\mathrm{S}}(B, C'')}$ and $\overline{\mathrm{HP}_{\mathrm{F}}(B, C'')}$, respectively.

Suppose $\mathbf{P}$ has the strong HC-SCP property and let H be the hamiltonian cycle that witnesses this. We divide the Type 1 and Type 2 blocks that partition the chains in H into two subclasses. Let $B = \{C_i, C_{i+1}\}$ be a Type 1 block, where C_i is comes before C_{i+1} in H. We call B *up-down*, or UD, if the H traverses C_i from the least element to the greatest element (and hence traverses C_{i+1} from greatest element to least element). Otherwise we call B *down-up*, or DU. Now suppose B is Type 2. We call B *up*, or U, if H traverses B from the least element to the greatest element. Otherwise we call B *down*, or D. For an example, see Figure 3.12. The block types there are: B_1 and B_5 are UD; B_2 and B_4 are U; B_3 is DU.

Theorem 3.5.7. *Suppose* $\mathbf{P}$ *has the strong HC-SCP property, let* H *be the hamiltonian cycle that witnesses this, and let* $C_1, C_2, \ldots, C_w$ *be the order of the chains encountered by* H*. Let* $B = \{C', C\}$ *be a Type 1 block. Then* $\mathbf{P} \times B$ *has a hamiltonian path that parses into symmetric chains such that either (1) the end of the path are* $S(C_1, C')$ *and* $S(C_1, C)$ *and the path avoids the edge from* $S(C_1, C')$ *to* $S(C_1, C)$*, or (2) the ends of the path are* $\overline{S}(C_1, C')$ *and* $\overline{S}(C_1, C)$ *and the path avoids the edge from* $\overline{S}(C_1, C')$ *to* $\overline{S}(C_1, C)$*.*

Proof. Let $B_1, B_2, \ldots, B_s$ be the blocks that partition the chains in H. We may assume that B_1 is UD or U and prove condition (1). If instead B_1 is DU or D, then condition (2) holds by applying this exact proof to $\mathbf{P}^D$. With this in mind, we construct the desired hamiltonian path by visiting vertices in the following order: $B_1 \times C'$, $B_2 \times C'$, $\ldots$, $B_s \times C'$, $B_s \times C$, $B_{s-1} \times C$, $\ldots$, $B_1 \times C$.

The manner in which we visit the vertices in the individual cartesian products depends on the type of the block B_i. First suppose B_i is Type 1. If $B_i = \{C_i, C_{i+1}\}$ is UD, then we traverse $B_i \times C'$ with $\mathrm{HP_S}(B_i, C')$ from $S(C_i, C')$ to $S(C_{i+1}, C')$, and we traverse $B_i \times C$ with $\mathrm{HP_S}(B_i, C)$ from $S(C_{i+1}, C)$ to $S(C_i, C)$. If B_i is DU, then we traverse $B_i \times C'$ with $\mathrm{HP_F}(B_i, C')$ from $F_{\mathrm{H_2}}(C_i, C')$ to $F_{\mathrm{H_2}}(C_{i+1}, C')$, and we traverse $B_i \times C$ with $\mathrm{HP_F}(B_i, C)$ from $F_{\mathrm{H_2}}(C_{i+1}, C)$ to $F_{\mathrm{H_2}}(C_i, C)$.

Next suppose B_i is Type 2, and hence a two-element chain. We know from Lemma 3.5.4 that the cartesian product of any chain and $\mathbf{2}$ is Type $\mathrm{H_2}$. Therefore, $B_i \times C'$ has a hamiltonian path with ends $S(B_i, C')$ and $F_{\mathrm{H_2}}(B_i, C')$, which we denote $H_{C'}$, and $B_i \times C$ has a hamiltonian path with ends $S(B_i, C)$ and $F_{\mathrm{H_2}}(B_i, C)$, which we denote H_C. For our construction, if B_i is U, then we traverse $B_i \times C'$ from $S(B_i, C')$ to $F_{\mathrm{H_2}}(B_i, C')$ with $H_{C'}$, and we traverse $B_i \times C$ from $F_{\mathrm{H_2}}(B_i, C)$ to $S(B_i, C)$ with H_C. If B_i is D, then we traverse $B_i \times C'$ from $F_{\mathrm{H_2}}(B_i, C')$ to $S(B_i, C')$ with $H_{C'}$, and we traverse $B_i \times C$ from $S(B_i, C)$ to $F_{\mathrm{H_2}}(B_i, C)$ with H_C.

We have partitioned the vertices of $\mathbf{P} \times B$ into symmetric chains, and further we have described a sequence of paths that starts at $S(C_1, C')$, ends at $S(C_1, C)$, and avoids the edge from $S(C_1, C')$ to $S(C_1, C)$. It remains to check that the traversals we have described above can be linked with edges in the cover graph of $\mathbf{P} \times B$ to create a hamiltonian path. To this end, the following observations are sufficient:

- Suppose the traversal of $B_i \times C'$ ends at (x_1, y_1) and the traversal of $B_{i+1} \times C'$ starts (x_2, y_2), for $i \in [s-1]$. Then x_1x_2 is an edge in H.

- Suppose the traversal of $B_{i+1} \times C$ ends at (x_1, y_1) and the traversal of $B_i \times C$ starts at (x_2, y_2), for $i \in [s-1]$. Then x_1x_2 is an edge in H.

- Suppose the traversal of $B_s \times C'$ ends at (x_1, y_1) and the traversal of $B_s \times C$ starts at (x_1, y_1). Then x_1x_2 is an edge in the cover graph of B.

For the first two observations, it is helpful to note that the transition from a block of Type UD or U to a block of Type DU or D in H must happen at a chain of length at most two. Furthermore, there is a never a transition from a Type U block to a Type D block, or vice-versa (see Fact 3.5.11). □

We shall denote the hamiltonian path constructed in the proof of Theorem 3.5.7 by HP$(\mathbf{P}, B)$. Clearly $B \times \mathbf{P}$ has the same hamiltonian path. For an example of these paths, see Figure 3.12. We can now prove the main result of this subsection.

Theorem 3.5.8. *Suppose $\mathbf{P}_1$ and $\mathbf{P}_2$ are posets with the strong HC-SCP property, and let $\mathcal{C}$ be any chain in the hamiltonian cycle that witnesses this for $\mathbf{P}_1$. Further suppose that, for $\mathbf{P}_2$, this fact can be witnessed by a hamiltonian cycle H with chains $C_1, C_2, \ldots, C_{2s}$ whose blocks are all Type 1. Then $\mathbf{P}_1 \times \mathbf{P}_2$ has a hamiltonian path that parses into symmetric chains such that either (1) the ends of the path are $S(\mathcal{C}, C_1)$ and $S(\mathcal{C}, C_{2s})$ and the path avoids the edge from $S(\mathcal{C}, C_1)$ to $S(\mathcal{C}, C_{2s})$, or (2) the ends of the path are $\overline{S}(\mathcal{C}, C_1)$ and $\overline{S}(\mathcal{C}, C_{2s})$ and the path avoids the edge from $\overline{S}(\mathcal{C}, C_1)$ to $\overline{S}(\mathcal{C}, C_{2s})$.*

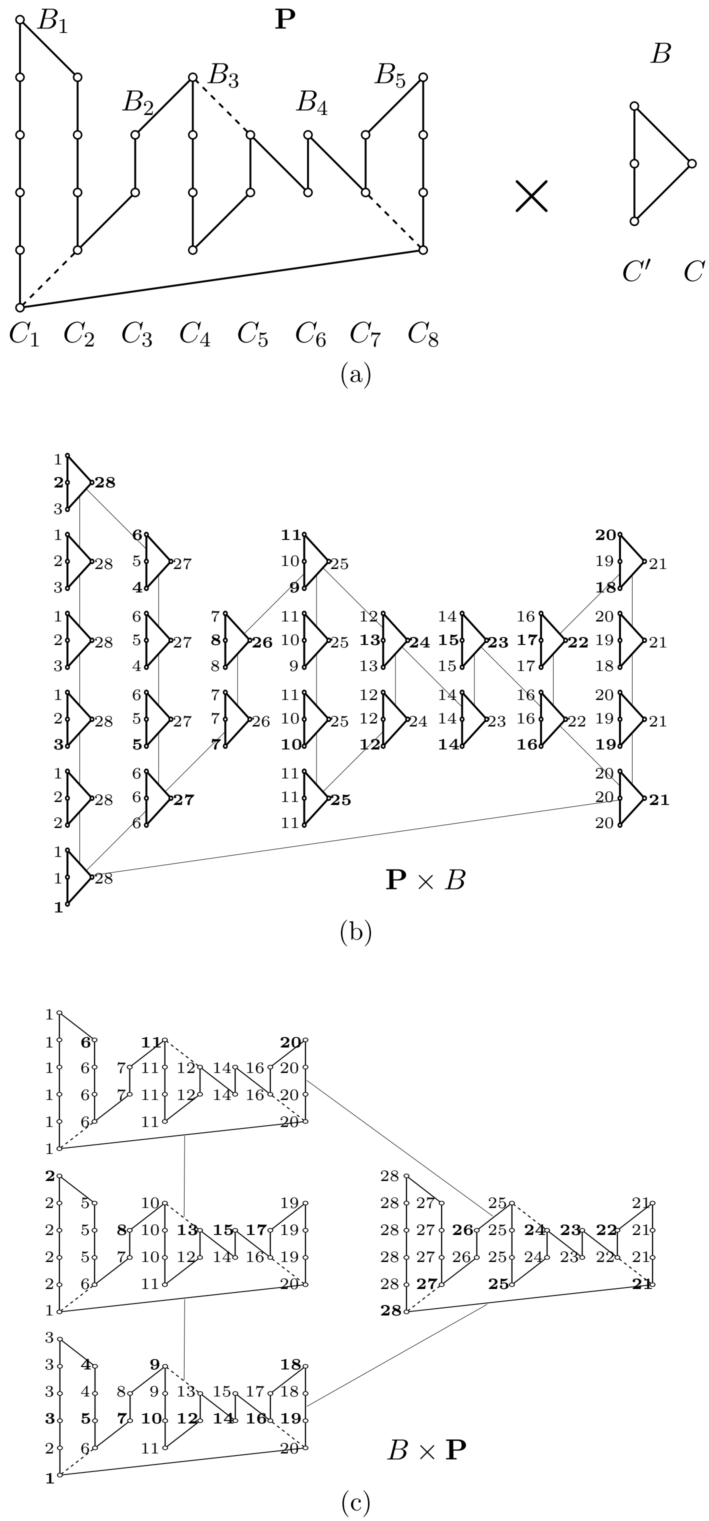


Figure 3.12: The numbers in Figure 3.12b and 3.12c are the symmetric chains. The first occurrence of a number in the hamiltonian cycle is bold.

Proof. We may assume that the block that contains $\mathcal{C}$ is either UD or U and prove condition (1). If instead this block is DU or D, then condition (2) holds by applying this proof with $\mathbf{P}_1^D$.

Let $B_i = \{C_{2i-1}, C_{2i}\}$ be the Type 1 blocks in H. We construct our hamiltonian path by traversing the vertices in the following order: HP$(\mathbf{P}_1, B_1)$, HP$(\mathbf{P}_1, B_2)$, $\ldots$, HP$(\mathbf{P}_1, B_s)$. Clearly every vertex of $\mathbf{P}_1 \times \mathbf{P}_2$ is used in exactly one of these paths, and every chain used is symmetric. We can link these paths together by noticing that the ends of each path are in the same copy of $\mathbf{P}_2$. In particular, we link these paths with the edges $\{S(\mathcal{C}, C_{2i}), S(\mathcal{C}, C_{2i+1})\}$ for all $i \in [s-1]$, to get a hamiltonian path in $\mathbf{P}_1 \times \mathbf{P}_2$. □

We shall denote the hamiltonian path constructed in Theorem 3.5.8 by HP$(\mathbf{P}_1, \mathbf{P}_2)$. The following corollary states that the cartesian product of two posets with the strong HC-SCP property has the HC-SCP property, subject to one further restriction on one of the two posets.

Corollary 3.5.9. *Suppose $\mathbf{P}_1$ and $\mathbf{P}_2$ are posets with the strong HC-SCP property. Further suppose that, for $\mathbf{P}_2$, this fact can be witnessed by a hamiltonian cycle whose blocks are all Type 1. Then $\mathbf{P}_1 \times \mathbf{P}_2$ has the HC-SCP property.*

Proof. There is an edge incident to the ends of HP$(\mathbf{P}_1, \mathbf{P}_2)$. □

In what follows we get rid of the extra restriction in Corollary 3.5.9.

3.5.3 Even width

If either poset in our cartesian product has a hamiltonian cycle that parses into symmetric chains such that every block of chains is Type 1, then we are done by Corollary 3.5.9, using that poset as $\mathbf{P}_2$. So in this section we assume that our posets do not have such a block structure. This allows us the following facts, which are easily verified.

Fact 3.5.10. Let $\mathbf{P}$ be a poset with the strong HC-SCP property and suppose this is witnessed by a hamiltonian cycle with a Type 2 block. Then the height of $\mathbf{P}$ is even.

Fact 3.5.11. Let $\mathbf{P}$ be a poset with the strong HC-SCP property and suppose this is witnessed by a hamiltonian cycle with a Type 2 block. Then all Type 2 blocks have the same type; they are either all U or all D.

Let $k \geq 2$ be a positive integer. Let $\mathbf{P}$ be a poset with ground set $\{a_1, b_1, a_2, b_2, \ldots, a_k, b_k\}$ and partial order consisting of comparabilities $a_i < b_i$ for each $i \in [k]$ and $a_{i+1} < b_i$ for each $i \in [k-1]$. Then we call $\mathbf{P}$ a *fence of length* k, denoted $\mathbb{F}_k$. The fence of length two is also called $\mathbf{N}$, and is depicted in Figure 3.13. If we add the comparability $a_1 < b_k$ to $\mathbb{F}_k$ we get the *crown of length* k, denoted $\mathbb{C}_k$.

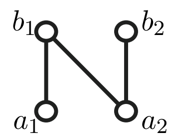


Figure 3.13: The poset $\mathbf{N}$

For the sake of notation, we introduce the following definitions. Let $\mathbf{P}$ be a poset with the strong HC-SCP property, witnessed by hamiltonian cycle H, where the chains in H are C_1, C_2, $\ldots$, C_w. Suppose the block structure of H must have at least one Type 2 block. We say $\mathbf{P}$ is in *standard position* when C_w is a Type 2 block of Type U. When $\mathbf{P}$ is in standard position, let w_1 be the least element of C_w, let w_2 the greatest, and let s_1 be the element of C_1 that is adjacent to w_2 in H.

Finally, before proving Lemma 3.5.13, we need one more fact. It follows from a careful reading of the proofs in Section 3.4.

Fact 3.5.12. Let $\mathbf{P}$ be a poset with the strong HC-SCP property and let $\mathbf{2} = \{0, 1\}$. Then $\mathbf{P} \times \mathbf{2}$ has two distinct hamiltonian paths that parse into symmetric chains and can be extended to hamiltonian cycles. One, as desired in Section 3.4, starts and ends in $\mathbf{P} \times \{0\}$. Let the starting and ending points of this path be $(x, 0)$ and $(y, 0)$, respectively. Then the second hamiltonian path uses the same chains as the first (in slightly different order), but starts at $(x, 1)$ and ends at $(y, 1)$. For example, the poset in Figure 3.7 has a hamiltonian path that uses the chains in the order 2, 1, 4, 3.

Lemma 3.5.13. *Let* $\mathbf{P}$ *be a poset with the strong HC-SCP property, witnessed by a hamiltonian cycle* H*. Suppose* $\mathbf{P}$ *is in standard position. Then* $\mathbf{P} \times \mathbf{N}$ *has a hamiltonian path that parses into symmetric chains with ends* (s_1, a_1) *and* (s_1, b_2)*.*

Proof. Let C_1 be the subposet of $\mathbf{N}$ induced by a_1 and b_1, and let C_2 be the subposet of $\mathbf{N}$ induced by a_2 and b_2. By Corollary 3.4.3, $\mathbf{P} \times C_1$ has the strong HP-SCP property. Let H_1 be the hamiltonian path that witnesses this, as constructed in Section 3.4. Notice that H_1 starts at (s_1, a_1) and ends at (w_2, a_1). However, since $\mathbf{2} \times \mathbf{2}$ is both Type H_1 and Type H_2, by Fact 3.5.1, we can amend H_1 so that it ends at (w_1, b_1) instead; the last two chains are $\{(w_1, a_1), (w_2, a_1), (w_2, b_1)\}$ and $\{(w_1, b_1)\}$. Call this amended hamiltonian path H_1'.

Similarly, using Fact 3.5.12, we can find a hamiltonian path in $\mathbf{P} \times C_2$ that parses into symmetric chains and starts at (s_1, b_2) and ends at (w_1, a_2). Call this path H_2'. Now we find the desired path in $\mathbf{P} \times \mathbf{N}$ by traversing H_1', following the edge from (w_1, b_1) to (w_1, a_2), then traversing H_2' in reverse. $\square$

The following corollary is an immediate consequence of Lemma 3.5.13, noting that $\mathbb{F}_{2k}$ is simply k $\mathbf{N}$'s concatenated together.

Corollary 3.5.14. *Let* k *be a positive integer, and let* $\mathbf{P}$ *be a poset with the strong HC-SCP property, witnessed by a hamiltonian cycle* H*. Suppose* $\mathbf{P}$ *is in standard position. Then* $\mathbf{P} \times \mathbb{F}_{2k}$ *has a hamiltonian path that parses into symmetric chains with ends* (s_1, a_1) *and* (s_1, b_{2k})*.*

Before we prove the main result of this subsection, we need a few more definitions. Let $\mathbf{P}$ be a poset with the strong HC-SCP property that is witnessed by hamiltonian cycle H. Let $C_1, C_2, \ldots, C_s$ be the chains in H and let $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ be the subsequence of Type 2 chains. Let $\mathbf{N}_j$ denote the pair $\{C_{i_{2j-1}}, C_{i_{2j}}\}$ for all $j \in \{1, 2 \ldots, \lfloor \frac{k}{2} \rfloor\}$. Define a *group of Type 1 blocks* to be a maximal sequence of consecutive chains of H such that each chain in the sequence is in a Type 1 block. These groups can arise in two different forms. We say a group G is *inside* if there is some j such that $C_{i_{2j-1}}$ and $C_{i_{2j}}$ are the chains immediately before and immediately after the chains of G in H, respectively. Otherwise we say G is *outside*. The group before C_{i_1} and the group after C_{i_k} are considered as distinct outside groups. In what follows, we denote the hamiltonian path constructed in Corollary 3.5.14 by $\mathrm{HP}(\mathbf{P}, \mathbb{F}_{2k})$, of simply $\mathrm{HP}(\mathbf{P}, \mathbf{N})$ when $k = 1$.

Theorem 3.5.15. *Suppose $\mathbf{P}_1$ and $\mathbf{P}_2$ are posets with the strong HC-SCP property. Further suppose that the width of $\mathbf{P}_2$ is even. Then $\mathbf{P}_1 \times \mathbf{P}_2$ has the HC-SCP property.*

Proof. Let H_1 and H_2 be hamiltonian cycles that witness the fact that $\mathbf{P}_1$ and $\mathbf{P}_2$ have the strong HC-SCP property, respectively. By Corollary 3.5.9, we may assume that H_1 and H_2 have blocks of Type 2. Since the width of $\mathbf{P}_2$ is even, we find that H_2 has an even number of Type 2 blocks.

Orient $\mathbf{P}_1$ so that it is in standard position; that is, if $D_1, D_2, \ldots, D_w$ are the chains of H_1, then $D_w = \{w_1, w_2\}$ is a Type 2 block of Type U. Suppose H_2 has only Type 2 blocks. Then $\mathbf{P}_2$ is isomorphic to $\mathbb{C}_{2k}$ for some positive integer k. Corollary 3.5.14 allows us to find $\mathrm{HP}(\mathbf{P}_1, \mathbb{F}_{2k})$, to which we add the edge from (s_1, a_1) to (s_1, b_{2k}) to find the desired hamiltonian cycle.

So we may assume H_2 has at least one Type 1 block. Let $C_1, C_2, \ldots, C_s$ be the chains in H_2, let $C_{i_1}, C_{i_2}, \ldots, C_{i_{2k}}$ be the subsequence of Type 2 chains, and let $\mathbf{N}_j$ denote the pair $\{C_{i_{2j-1}}, C_{i_{2j}}\}$ for all $j \in [k]$. We can now piece together the desired hamiltonian cycle. Suppose that we are either starting our construction or that we have some nonempty part of our cycle constructed. In the latter case, suppose we have visited exactly the points in $\mathbf{P}_1 \times \{C_1 \cup C_2 \cup \ldots C_t\}$ for some $t \in [s]$, where C_t is either the second Type 2 block of some $\mathbf{N}_j$ or the last chain in some outside group, and have ended in $\{s_1\} \times \mathbf{P}_2$. We then proceed as follows.

There are three cases to consider. First, suppose C_{t+1} is the first Type 2 block in $\mathbf{N}_{j+1}$, and suppose $\mathbf{N}_{j+1}$ does not have a group of Type 1 blocks inside of it. Then we add $\mathrm{HP}(\mathbf{P}_1, \mathbf{N}_{j+1})$ via the construction in Lemma 3.5.13 to our cycle.

Second, suppose C_{t+1} is the first Type 2 block in $\mathbf{N}_{j+1}$, but in this case $\mathbf{N}_{j+1}$ has a group of Type 1 blocks inside of it, G. Let $\mathbf{N}_{j+1}$ consist of the points a_1, b_1, a_2, and b_2, as in Figure 3.13, and let the blocks of G be B_1, B_2, $\ldots$, B_q. We now add points to our cycle in the following way: start by adding the first half of $\mathrm{HP}(\mathbf{P}_1, \mathbf{N}_{j+1})$ via the construction in Lemma 3.5.13 (there it was called H_1'). Follow the edge from (w_1, b_1) to (w_1, z_1), where z_1 is the least element of B_1. Next add $\mathrm{HP}(\mathbf{P}_1, B_1)$, followed by $\mathrm{HP}(\mathbf{P}_1, B_2)$, $\ldots$, followed by $\mathrm{HP}(\mathbf{P}_1, B_q)$ to our cycle, via the construction in Theorem 3.5.7 (notice that each of these paths traverse the chains of H_1 in the order D_w, D_1, D_2, $\ldots$, D_{w-1}, and back). Follow the edge from (w_1, z_2) to (w_1, a_2). Finally, add the second half of $\mathrm{HP}(\mathbf{P}_1, \mathbf{N}_{j+1})$ via the construction in Lemma 3.5.13 (there is was called H_2') in reverse.

Last, suppose C_{t+1} is the first chain in an outside group of Type 1 blocks, G. Let the blocks of G be B_1, B_2, $\ldots$, B_q. Then we add $\mathrm{HP}(\mathbf{P}_1, B_1)$, followed by $\mathrm{HP}(\mathbf{P}_1, B_2)$, $\ldots$, followed by $\mathrm{HP}(\mathbf{P}_1, B_q)$ to our cycle, via the construction in Theorem 3.5.7.

In each case we return to the initial conditions; we have visited exactly the points in $\mathbf{P}_1 \times \{C_1 \cup C_2 \cup \ldots C_{t'}\}$, where $C_{t'}$ is either the second Type 2 block of some $\mathbf{N}_j$ or the last chain in some outside group, and have ended in $\{s_1\} \times \mathbf{P}_2$. Therefore we can continue this process until all of the points in $\mathbf{P}_1 \times \mathbf{P}_2$ have been visited.

We now have a partition of the points in $\mathbf{P}_1 \times \mathbf{P}_2$ into symmetric chains. It remains to show that we can link the hamiltonian paths produced in the construction above in order to obtain a hamiltonian cycle. But this is easy, as all such transitions occur in $\{s_1\} \times \mathbf{P}_2$ and $\mathbf{P}_2$ has the strong HC-SCP property. $\square$

An example of this construction can be found in Figure 3.15, where it is combined with the techniques described in the next subsection.

3.5.4 Odd width

Just as we focused on even fences and even crowns in the previous subsection, we focus on odd fences and odd crowns here.

Lemma 3.5.16. *Let j be a positive integer. Then $\mathbb{F}_3 \times \mathbb{F}_{2j+1}$ has a hamiltonian path that parses into symmetric chains with ends (a_1, a_1) and (b_3, a_1).*

Proof. We give an explicit construction of the chains that are used. First, we cover $\{a_1, b_1, a_2, b_2\} \times \{a_1, b_1, a_2, b_2\}$, which is isomorphic to $\mathbf{N} \times \mathbf{N}$, by the eight chains

$$\{(a_1, a_1), (a_1, b_1), (b_1, b_1)\}, \quad \{(b_1, a_1)\}, \quad \{(a_2, a_1), (b_2, a_1), (b_2, b_1)\}, \quad \{(a_2, b_1)\},$$

$$\{(a_2, a_2), (b_2, a_2), (b_2, b_2)\}, \quad \{(a_2, b_2)\}, \quad \{(b_1, b_2), (b_1, a_2), (a_1, a_2)\}, \quad \{(a_1, b_2)\}.$$

We then cover all elements in $\{a_1, b_1, a_2, b_2\} \times \{a_3, b_3, a_4, b_4, \ldots, a_{2j}, b_{2j}\}$, using this same strategy, ending at the point (a_1, b_{2j}). We cover the remaining elements in $\{a_1, b_1, a_2, b_2\} \times \mathbb{F}_{2j+1}$ with the four chains

$$\{(a_1, a_{2j+1}), (a_1, b_{2j+1}), (b_1, b_{2j+1})\}, \quad \{(b_1, a_{2j+1})\},$$

$$\{(a_2, a_{2j+1}), (a_2, b_{2j+1}), (b_2, b_{2j+1})\}, \quad \{(b_2, a_{2j+1})\}.$$

Next we use the chains $\{(a_3, a_{2j+1}), (a_3, b_{2j+1}), (b_3, b_{2j+1})\}$ and $\{(b_3, a_{2j+1})\}$. Finally, we use the chains $\{(b_3, b_i), (a_3, b_i), (a_3, a_i)\}$ and $\{(b_3, a_i)\}$ for $i = 2j, 2j-1, \ldots, 1$, in that order, to finish the desired hamiltonian path. $\square$

The following corollary is an immediate consequence of Lemma 3.5.16 and Lemma 3.5.13, noting that $\mathbb{F}_{2k+1}$ is simply $\mathbb{F}_3$ followed by $k-1$ $\mathbf{N}$'s concatenated together.

Corollary 3.5.17. *Let k and j be positive integers. Then $\mathbb{F}_{2k+1} \times \mathbb{F}_{2j+1}$ has a hamiltonian path that parses into symmetric chains with ends (a_1, a_1) and (b_{2k+1}, a_1).*

Before proving the main result of this subsection, we require two more technical lemmas.

Lemma 3.5.18. *Let* $\mathbf{P}$ *be a poset with the strong HC-SCP property, witnessed by hamiltonian cycle* H*, and suppose* H *has at least one Type 2 block. Then each group* G *(inside or outside) in* H *has a block* $B = \{C, C'\}$ *such that* $\{|C|, |C'|\} = \{2, 4\}$.

Proof. Let $B_1, B_2, \ldots, B_t$ be the blocks in G. Suppose there are blocks $B_i = \{C_{i_1}, C_{i_2}\}$ and $B_k = \{C_{k_1}, C_{k_2}\}$ such that $|C_{i_1}|$ is a multiple of four and $|C_{k_1}|$ is not, for some $1 \leq i < k \leq t$. Then it is easy to verify that there is a block B_j, where $i < j < k$, that has a two-element chain.

Let C_1 and C_2 be the chains in H that occur immediately before and immediately after the chains of G, respectively. By the maximality of G, both C_1 and C_2 are Type 2 blocks (if H has just one block of Type 2, then $C_1 = C_2$.) If B_s, for any $s \in [t]$, has a chain of length two, then we are done. So we may assume the first chain in B_1 has length four and the second has length six. Similarly, we may assume the first chain in B_t has length six and the second has length four. Therefore, we are done by the argument in the preceding paragraph. □

Lemma 3.5.19. *Let* $\mathbf{P}$ *be a poset with a hamiltonian cycle that parses into symmetric chains* $C_1, C_2, \ldots, C_t$ *with* t *odd and* $\mathbf{P}$ *in standard position. Let* $B = \{C, C'\}$ *be a Type 1 block such that* C *is* $a_1 < a_2 < a_3 < a_4$ *and* C' *is* $b_1 < b_2$*. Then* $\mathbf{P} \times B$ *has a hamiltonian path* H *that parses into symmetric chains that starts at* (w_1, a_4) *and ends at* (s_1, b_2).

Proof. Since $\mathbf{P}$ is in standard position, we know that C_t is U. By Fact 3.5.11, all Type 2 blocks in this Lemma are also U. For the sake of notation, we refer to the block containing C_1 as B_1, and we refer to the points in C_i as $c_{i_1} < c_{i_2} < \cdots < c_{i_k}$ where k is the height of C_i and $1 \leq i \leq t$ (thus $w_1 = c_{t_1}$ and $w_2 = c_{t_2}$). We proceed by proving two claims and then by showing that these claims imply the Lemma.

Claim 3.5.20. Suppose all blocks in $\mathbf{P}$ are Type 1 except for the last block, C_t. If B_1 is UD or U, then there is a hamiltonian path H' that parses into symmetric chains, starts at (c_{t_1}, a_4), and ends at (c_{1_1}, b_2). If B_1 is DU, then there is a hamiltonian path H'' that parses into symmetric chains, starts at (c_{t_1}, a_4), and ends at (c_{1_k}, b_2).

We prove Claim 3.5.20 using induction on t. If $t = 1$, in which case $C_1 = C_t$ (recall that we defined $\mathbf{2}$ to be hamiltonian), then H' has the chains $\{(c_{1_1}, a_4), (c_{1_1}, a_3), (c_{1_1}, a_2)\}$, $\{(c_{1_1}, a_1), (c_{1_2}, a_1), (c_{1_2}, a_2), (c_{1_2}, a_3), (c_{1_2}, a_4)\}$, $\{(c_{1_2}, b_2), (c_{1_2}, b_1), (c_{1_1}, b_1)\}$, and $\{(c_{1_1}, b_2)\}$. Now assume $t \geq 3$ (by the assumption that all blocks other than the last are Type 1, we know t is odd).

Suppose B_1 is UD. Then $|C_1| = 2$ and $|C_2| = 4$. Therefore, the block containing C_3 is also UD or U. By the inductive hypothesis, there is a hamiltonian path that parses into symmetric chains from (c_{t_1}, a_4) to (c_{3_1}, b_2). We complete H' in the following manner: use the edge $\{(c_{3_1}, b_2), (c_{2_1}, b_2)\}$, traverse $\mathrm{HP}_{\mathrm{F}}(B, C_2)$ in reverse to (c_{2_1}, a_4), use the edge $\{(c_{2_1}, a_4), (c_{1_1}, a_4)\}$, and traverse $\mathrm{HP}_{\mathrm{F}}(B, C_1)$ to (c_{1_1}, b_2).

Now suppose B_1 is DU. Then $|C_1| = 4$ and $|C_2| \in \{2, 6\}$. Assume the block containing C_3 is DU. By the inductive hypothesis, there is a hamiltonian path that parses into symmetric chains from (c_{t_1}, a_4) to (c_{3_k}, b_2). We complete H'' in the following manner: use the edge $\{(c_{3_k}, b_2), (c_{2_k}, b_2)\}$, traverse $\overline{\mathrm{HP}_{\mathrm{S}}(B, C_2)}$ in reverse to (c_{2_k}, a_4), use the edge $\{(c_{2_k}, a_4), (c_{1_k}, a_4)\}$, and traverse $\overline{\mathrm{HP}_{\mathrm{S}}(B, C_1)}$ to (c_{1_k}, b_2). Now assume the block containing C_3 is UD or U. Then $|C_2| = 2$. By the inductive hypothesis there is a hamiltonian path that parses into symmetric chains from (c_{t_1}, a_4) to (c_{3_1}, b_2). We complete H'' by using the edge $\{(c_{3_1}, b_2), (c_{2_k}, b_2)\}$ and then by following the same steps as in the preceding case.

Claim 3.5.21. Suppose all blocks in $\mathbf{P}$ are Type 1 except for the last block, C_t. If B_1 is UD or U, then there is a hamiltonian path H' that parses into symmetric chains, starts at (c_{t_2}, b_2), and ends at (c_{1_1}, a_1). If B_1 is DU, then there is a hamiltonian path H'' that parses into symmetric chains, starts at (c_{t_2}, b_2), and ends at (c_{1_k}, a_1).

We prove Claim 3.5.21 using induction on t. If $t = 1$, in which case $C_1 = C_t$, then H' has the chains $\{(c_{1_2}, b_2), (c_{1_1}, b_2), (c_{1_1}, b_1)\}$, $\{(c_{1_2}, b_1)\}$, $\{(c_{1_2}, a_1), (c_{1_2}, a_2), (c_{1_2}, a_3)\}$, and $\{(c_{1_2}, a_4), (c_{1_1}, a_4), (c_{1_1}, a_3), (c_{1_1}, a_2), (c_{1_1}, a_1)\}$. Now assume $t \geq 3$.

Suppose B_1 is UD. As above, $|C_1| = 2$, $|C_2| = 4$, and the block containing C_3 is also UD or U. By the inductive hypothesis, there is a hamiltonian path that parses into symmetric chains from (c_{t_2}, b_2) to (c_{3_1}, a_1). We complete H' in the following manner: use the edge $\{(c_{3_1}, a_1), (c_{2_1}, a_1)\}$, traverse $\mathrm{HP_S}(B, C_2)$ to (c_{2_1}, b_1), use the edge $\{(c_{2_1}, b_1), (c_{1_1}, b_1)\}$, and traverse $\mathrm{HP_S}(B, C_1)$ in reverse to (c_{1_1}, a_1).

Now suppose B_1 is DU. As above, $|C_1| = 4$ and $|C_2| \in \{2, 6\}$. Assume the block containing C_3 is DU. By the inductive hypothesis, there is a hamiltonian path that parses into symmetric chains from (c_{t_2}, b_2) to (c_{3_k}, a_1). We complete H'' in the following manner: use the edge $\{(c_{3_k}, a_1), (c_{2_k}, a_1)\}$, traverse $\overline{\mathrm{HP_F}(B, C_2)}$ to (c_{2_k}, b_1), use the edge $\{(c_{2_k}, b_1), (c_{1_k}, b_1)\}$, and traverse $\overline{\mathrm{HP_F}(B, C_1)}$ to (c_{1_k}, a_1). Now assume the block containing C_3 is UD or U. Then, as above, $|C_2| = 2$. By the inductive hypothesis, there is a hamiltonian path that parses into symmetric chains from (c_{t_2}, b_2) to (c_{3_1}, a_1). We complete H'' by using the edge $\{(c_{3_1}, a_1), (c_{2_k}, a_1)\}$ and then by following the same steps as in the preceding case.

We are now prepared to complete the proof of Lemma 3.5.19. Since t is odd there are an odd number of Type 2 blocks in the hamiltonian cycle of $\mathbf{P}$. Label these Type 2 blocks in the opposite order from the order they appear in H; that is, we have the sequence $C_t = \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{2j-1}$. Define $\mathcal{B}_i$ to be the set of elements in a Type 1 block between $\mathcal{C}_{i+1}$ and $\mathcal{C}_i$. Then define $\mathcal{D}_i = \mathcal{C}_i \cup \mathcal{B}_i$. It is clear that the $\mathcal{D}_i$ partition the elements of $\mathbf{P}$.

We construct H as follows: for odd i we traverse $\mathcal{D}_i \times B$ using the method of Claim 3.5.20, and for even i we traverse $\mathcal{D}_i \times B$ using the method of Claim 3.5.21. We need to verify that these traversals can be linked by edges in $\mathbf{P} \times B$. Going from $\mathcal{D}_{2i-1}$ to $\mathcal{D}_{2i}$ for $i \in [j-1]$ is straightforward given the statements of the claims. Going from $\mathcal{D}_{2i}$ to $\mathcal{D}_{2i+1}$ for $i \in [j-1]$ needs a quick justification since the path through $C_t \times B$ behaves differently than the path through $\mathcal{C}_{2i+1} \times B$. Let $\mathcal{C}_{2i+1}$ be the chain $c_1 < c_2$. Now we use the elements of $\mathcal{C}_{2i+1} \times B$ as follows: from $\mathcal{D}_{2i}$ we follow the edge to (c_2, a_1), then we use the chains $\{(c_2, a_1), (c_2, a_2), (c_3, a_3)\}$, $\{(c_2, a_4), (c_1, a_4), (c_1, a_3), (c_1, a_2), (c_1, a_1)\}$, $\{(c_1, b_1), (c_2, b_1), (c_2, b_2)\}$, $\{(c_1, b_2)\}$. Now we can continue our path through $\mathcal{D}_{2i+1}$ as before.

Since we have an odd number of $\mathcal{D}_i$, it is clear that the last element used in H is (s_1, b_2), as desired. See Figure 3.14 for an example. $\square$

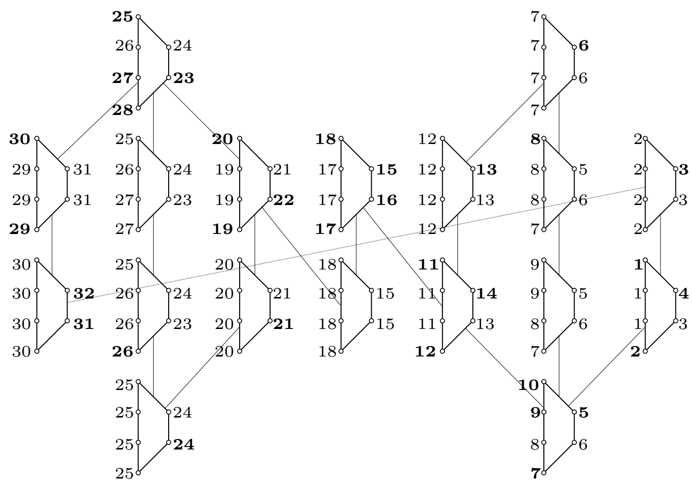


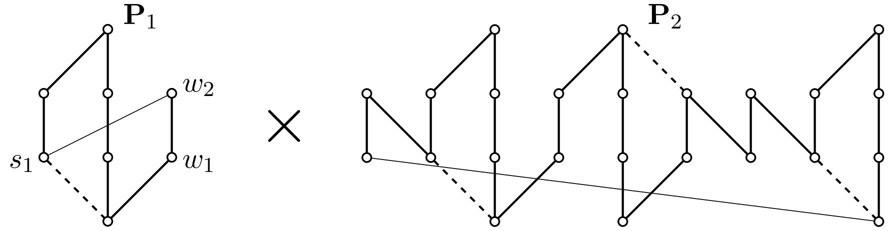
Figure 3.14: An example of $\mathbf{P} \times B$ from the proof of Lemma 3.5.19. The numbers are the symmetric chains in $\mathbf{P} \times B$. The first occurrence of a number in the hamiltonian path is bold.

Corollary 3.5.22 states that we can find a hamiltonian path in $\mathbf{P} \times B$ under the same conditions as Lemma 3.5.19 with the lone exceptions being the orientation of B (perhaps the chain of length two appears first) and ends of the path. We omit the proof of Corollary 3.5.22 as it is entirely analogous to the proof of Lemma 3.5.19.

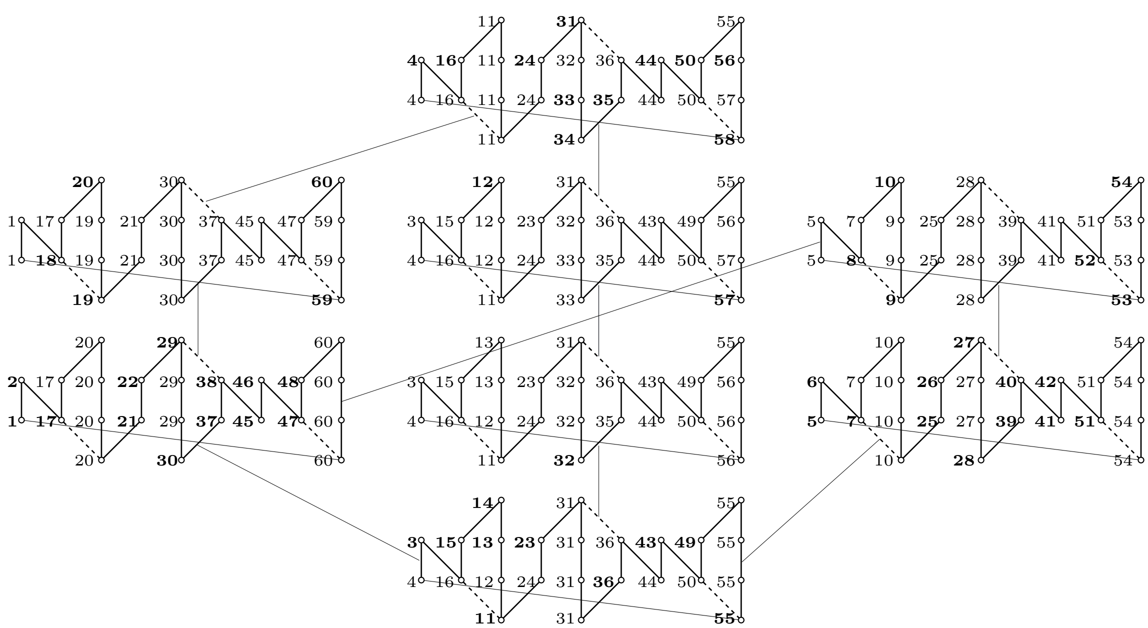
Corollary 3.5.22. *Let* $\mathbf{P}$ *be a poset with a hamiltonian cycle that parses into symmetric chains* $C_1, C_2, \ldots, C_t$ *with* t *odd and* $\mathbf{P}$ *in standard position. Let* $B = \{C, C'\}$ *be a Type 1 block with* $\{|C|, |C'|\} = \{2, 4\}$. *If* C *is* $a_1 < a_2 < a_3 < a_4$ *and* C' *is*

$b_1 < b_2$, then $\mathbf{P} \times B$ has a hamiltonian path H that parses into symmetric chains that starts at (w_1, a_1) and ends at (s_1, b_1). If C is $a_1 < a_2$ and C' is $b_1 < b_2 < b_3 < b_4$, then $\mathbf{P} \times B$ has a hamiltonian path H that parses into symmetric chains that starts at (w_1, a_2) and ends at (s_1, b_4) and another that starts at (w_1, a_1) and ends at (s_1, b_1).

When taken together with Theorem 3.5.15, Theorem 3.5.23 finishes the proof that the strong HC-SCP property is weakly closed under cartesian products. For a visual representation of the details in Theorem 3.5.23, see Figure 3.15. In what follows, the hamiltonian path constructed in Corollary 3.5.17 will be denoted $\mathrm{HP}(\mathbb{F}_{2k+1}, \mathbb{F}_{2j+1})$.



(a) $\mathbf{P}_1$ is in standard position. The first three chains of $\mathbf{P}_2$ constitute $\mathbf{P}'$ in the proof of Theorem 3.5.23. The second and third chains of $\mathbf{P}'' = \mathbf{P}_2 - \mathbf{P}'$ are an inside group, and the last two chains of $\mathbf{P}''$ are an outside group.



(b)

Figure 3.15: The numbers in Figure 3.15b are the symmetric chains in $\mathbf{P}_1 \times \mathbf{P}_2$. The first occurrence of a number in the hamiltonian cycle is bold.

Theorem 3.5.23. *Suppose* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are posets with the strong HC-SCP property. Further suppose that the width of* $\mathbf{P}_2$ *is odd. Then* $\mathbf{P}_1 \times \mathbf{P}_2$ *has the HC-SCP property.*

Proof. Let H_1 and H_2 be hamiltonian cycles that witness the fact that $\mathbf{P}_1$ and $\mathbf{P}_2$ have the strong HC-SCP property, respectively. By Corollary 3.5.9, we may assume that H_1 and H_2 have blocks of Type 2. By Theorem 3.5.15, we may assume the width of $\mathbf{P}_1$ is odd as well; otherwise, we could exchange the roles of $\mathbf{P}_1$ and $\mathbf{P}_2$. Thus, we find that H_1 and H_2 have an odd number of Type 2 blocks.

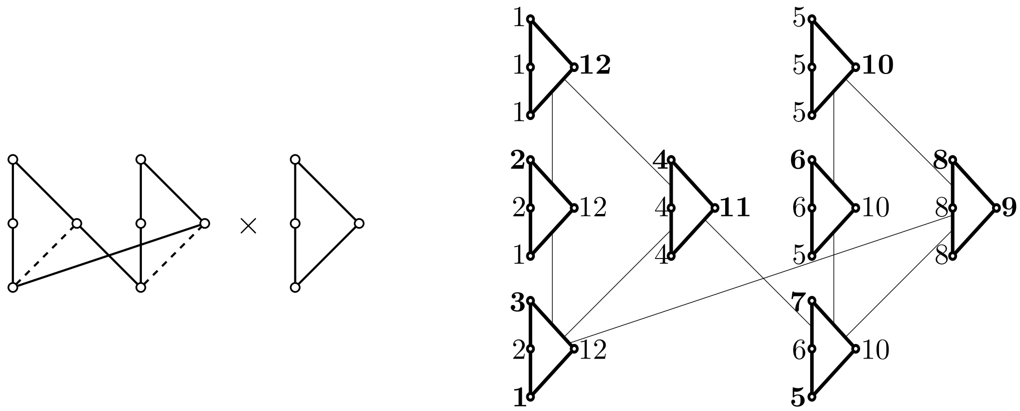
Suppose all blocks in H_1 and H_2 are Type 2. Then there are positive integers k and j such that $\mathbf{P}_1$ and $\mathbf{P}_2$ are isomorphic to $\mathbb{C}_{2k+1}$ and $\mathbb{C}_{2j+1}$, respectively. Corollary 3.5.17 allows us to find $\mathrm{HP}(\mathbb{F}_{2k+1}, \mathbb{F}_{2j+1})$, to which we add the edge from (a_1, a_1) to (b_{2k+1}, a_1) to find the desired hamiltonian cycle.

Without loss of generality, H_2 has a block of Type 1. Orient H_2 so that its first block, B_1, is Type 2 and U and its second block, B_2, is Type 1. Therefore the group G containing B_2 is an inside group. Further, orient $\mathbf{P}_1$ so that it is in standard position. We then construct our desired hamiltonian cycle in the following way. Start by using the strategy developed in the proof of Theorem 3.5.15. Continue to use this strategy until a Type 1 block $B = \{C, C'\}$ in G with $\{|C|, |C'|\} = \{2, 4\}$ is reached. Such a block is guaranteed by Lemma 3.5.18. Traverse $\mathbf{P}_1 \times B$ according to the method demonstrated in the proof of Lemma 3.5.19 or Corollary 3.5.22, depending on the orientation of B. At this stage, we have visited all points in $\mathbf{P}_1 \times \mathbf{P}'$, where $\mathbf{P}'$ consists of all points in H_2 from B_1 to B. Let $\mathbf{P}$” $= \mathbf{P}_2 - \mathbf{P}'$. Notice that $\mathbf{P}$” has even width. Therefore, we use the strategy developed in the proof of Theorem 3.5.15 on $\mathbf{P}_1 \times \mathbf{P}$” to finish off our desired hamiltonian cycle. $\square$

3.6 Strong closure

The question “Does the cartesian product of any two posets with the strong HC-SCP property have the strong HC-SCP property?” remains open. In this section we discuss why the method described in Section 3.5 fails to resolve the question.

Consider the two posets and their cartesian product in Figure 3.16. The method described in Section 3.5 produces the hamiltonian cycle shown in Figure 3.16b.



(a) (b)

Figure 3.16: The numbers in Figure 3.16b are the symmetric chains. The first occurrence of a number in the hamiltonian cycle is bold.

Let H be the hamiltonian cycle depicted in Figure 3.16b. We redraw H in Figure 3.17 to get a better look at the symmetric chain partition that H parses into. Since the height of the cartesian product is odd, all blocks must be Type 1 if H were to witness the strong HC-SCP property. However, C_4 cannot form a Type 1 block with either C_3 or C_5 due to the edges missing from the product.

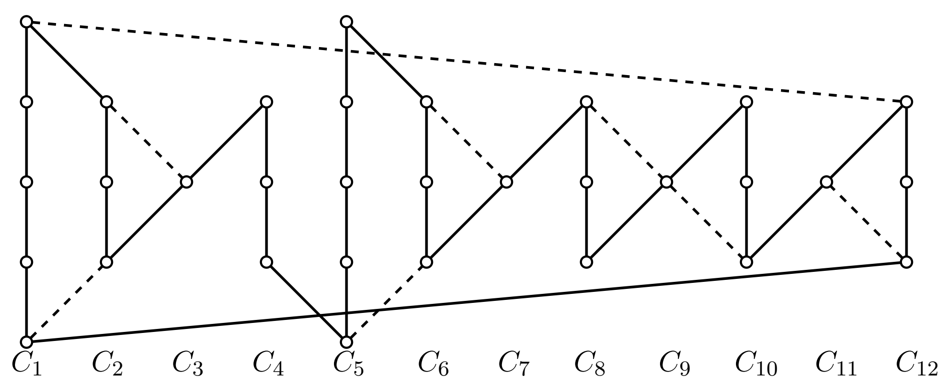


Figure 3.17

3.7 Connections with other cycle and path problems

The motivation for this line of research was twofold. First, we were hoping this investigation would lead to progress on the well-known "Middle Two Levels" conjecture. Although the origins of this problem are not completely clear, we can date it back to a visit to Prague in the summer of 1983 where the late Ivan Havel showed W. T. Trotter a reprint in which he had posed the problem. Regrettably, we have not been able to identify the specific paper in question.

Conjecture 1 (Middle Two Levels)**.** For every $n \geq 1$, the bipartite graph formed by the middle two levels of the subset lattice $\mathcal{B}(2n+1)$ is hamiltonian.

Let c be the length of the longest cycle in $\mathcal{B}(2n+1)$. A hamiltonian cycle in the middle two levels of $\mathcal{B}(2n+1)$ would visit all $N = 2\binom{2n+1}{n}$ vertices. So, Conjecture 1 states that $c/N = 1$. Felsner and Trotter [20] were the first to show that $\mathcal{B}(n)$ has a cycle that uses a positive fraction of the N vertices, when they proved $c/N \geq 1/4$. Savage and Winkler [43] made the next significant improvement when they showed $c/N \geq .839$, and later Savage and Shields [44] showed $c/N \geq .86$. The best known result is due to Johnson [28], who proved that $c/N = 1 - o(1)$.

Second, we were hoping to make progress on the question of whether $\mathcal{B}(n)$ has a monotone hamiltonian path. A *monotone hamiltonian path* is a listing $S_1, S_2, \ldots, S_t$ (where $t = 2^n$) of all subsets of $[n]$ so that (1) the order of the sets listed induces a hamiltonian path in the cover graph of $\mathcal{B}(n)$, (2) S_1 is the empty set, and (3) if $1 \leq i < j \leq t$ and S_j is a subset of S_i, then $j = i + 1$. Neither goal has (at least, as of now) been attained.

CHAPTER IV

CONCLUSION

In this dissertation, we have explored two topics relating to cover graphs of posets. In the first, we showed that there is a relationship between dimension and planarity when we restrict our attention to posets of bounded height. In particular, we provided a proof of a constant c_h, depending only on h, that bounds the dimension of posets with planar cover graphs and height at most h, and thus bounds the dimension of planar posets with height at most h as well. However, we suspect there is much room to improve the known bounds on c_h. While the arguments in Chapter 2 can surely be tweaked to yield somewhat better results, an entirely new argument may be required to prove near-optimal bounds on c_h. Other interesting open problems in this area include determining which posets are subposets of planar posets and finding a short proof of Kuratowski's theorem using dimension theory.

The second topic dealt with the cartesian product of posets with certain hamiltonian cycles in their cover graphs. As discussed in Chapter 3, this work was partly motivated by connections to the Middle Two Levels conjecture and the monotone hamiltonian path problem.

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VITA

Noah Streib was born in La Jolla, California, spent most of his next ten years in Shaker Heights, Ohio, then moved to Ada, Ohio where he attended middle school and high school. In 2006, he earned a B.A. in Applied Mathematics and a minor in Economics from Oberlin College, where he graduated with high honors, received the Rebecca Cary Orr Memorial Prize in Mathematics from the Oberlin College Department of Mathematics, and earned membership in Phi Beta Kappa. While there, he played all four years as a member of the Oberlin College soccer team, two years as a member of the Oberlin College baseball team, and one year as a Flying Horsecow as a member of the ultimate frisbee team. During the spring semester of his junior year, he studied in Budapest, Hungary in the Budapest Semesters in Mathematics program. Streib began his tenure in the Algorithms, Combinatorics, and Optimization program at Georgia Tech in 2006, where he received the Robert Price Fellowship from the School of Mathematics. In recognition of his teaching, he was the 2011 School of Math departmental nominee for the CETL/BP Outstanding TA Award as a lead instructor, the Institute's highest honor for a graduate teaching assistant. In graduate school Streib continued his involvement in organized athletics as a member of the Tribe, Georgia Tech's ultimate frisbee team.